P17586

BULLETIN

OF THE

CALCUTTA MATHEMATICAL SOCIETY

CALCUTTA UNIVERSITY CENTRAL LIBRARY

287

BOARD OF EDITORS

N. M. BASU, D.Sc. F. W. LEVI, DR.PHIL.NAT. C. V. H. RAO, M.A.
N. R. SEN, D.Sc., PH.D. M. R. SIDDIQI, M.A., PH.D.
S. GHOSH, D.Sc. (*Secretary*).

VOL. 35 MARCH, 1943 No. 1

CONTENTS

PAGE

1. Modern Algebra—By F. W. Levi 1
2. Some Infinite Integrals involving Bessel Functions—By H. C. Gupta 7
3. Stresses due to Forces and Couples acting in the Interior of a Slab placed on a Rigid Foundation—By Bibhutibhusan Sen 13
4. A Note on Sonine's Polynominals I—By K. Basu ... 21

Review 33

Accounts 35

CALCUTTA UNIVERSITY PRESS

1943

All correspondence with the Society, subscriptions to the Bulletin, admission fees and annual contributions of members are to be sent to the *Secretary, Calcutta Mathematical Society, 92, Upper Circular Road, Calcutta.*

Papers intended for publication in the Bulletin of the Society, and all Editorial Correspondence, should be addressed to the *Editorial Secretary, Calcutta Mathematical Society.*

The publications of the Calcutta Mathematical Society may be purchased direct from the Society's office, or from its agents—Messrs. Bowes & Bowes, Booksellers and Publishers, 1, Trinity Street, Cambridge, England.

NOTICE TO AUTHORS

The manuscript of each paper communicated for publication in this Journal should be legibly written (preferably type-written) on one side of the paper and should be accompanied by a short *abstract* of the paper.

References to literature in the text should be given, whenever possible, in chronological order, only the names of the authors and years of publication, in brackets, being given. They should be cited in full at the end of the paper, the author's name following in alphabetical order, and each reference is to be arranged in the following form, *viz.*, name or names of authors ; year of publication ; name of the journal (abbreviation) ; number of volume ; and lastly the page number. The following would be a useful illustration :

Wilson, B. M., 1922, *Proc. Lond. Math. Soc.* (2), 21, 235.

Authors of papers printed in the Bulletin are entitled to receive, free of cost, 25 separate copies of their communications. They can, however, by previous notice to the Secretary, ascertain whether it will be possible to obtain more copies even on payment of the usual charges.

BULLETIN

OF THE

CALCUTTA

MATHEMATICAL SOCIETY

CALCUTTA UNIVERSITY
CENTRAL LIBRARY

16

Vol. 35

1943

INDEX

		PAGE
Basu, K. ...	A Note on Sonine's Polynomials (I)	21
Do. ...	A Note on Sonine's Polynomials (II)	127
Bhattacharyya, A.	On a Measure of Divergence between two Statistical Populations defined by their Probability Distributions	99
Biswas. B. N. ...	Spontaneous Energy Fluctuation and Quantum Statistics	55
Chowdhury, A. C. ...	On a Generalisation of Reidemeister's Figure in a Web	77
Ghosh, S. ...	Stress Systems in rotating Aeolotropic Discs ...	61
Ghosh, N. N. ...	A Matrix-theory of Screws in Hyper-space ...	115
Gupta, H. C. ...	Some Infinite Integrals involving Bessel Functions...	7
Do. ...	Some Self-reciprocal Functions	67
Hameed, Asghar ...	On Mutually Self-polar Simplexes	43
Jabbar, M. A. ...	On S-groups	111
Levi, F. W. ...	Modern Algebra	1
Minakshisundaram, S.	On Expansion in Eigen-functions of Boundary Value Problems VI	81
Pillai, S. S. ...	Highly Abundant Numbers	141
Roy, Sunil Kumar ...	A Case of Polytropic Equilibrium Configuration with variable Angular Velocity	85
Sen, Bibhutibhusan	Stresses due to Forces and Couples acting in the Interior of a Slab placed on a Rigid Foundation...	13
Do. ...	Note on the Uniqueness of Solution of Problems of thin Plates bent by Normal Pressures ...	135
Sen, N. R. ...	On a Problem in Heat Conduction in a Sphere ...	131
Sinha, S. ...	Some Infinite Integrals involving Bessel Functions of Imaginary Arguments...	37
Sircar, H. ...	On Harmonic Functions	71

CALCUTTA UNIVERSITY CENTRAL LIBRARY

MODERN ALGEBRA*

BY

F. W. LEVI

It is a peculiar and even a dangerous thing to style a branch of Mathematics as "modern." Mathematical truth is expected to be everlasting, independent of time as well as of space, but "Modern" is what tallies with the taste and the fashion of these days; it bears the stamp of the present times and it is bound to pass away with them. For this reason I prefer in general the term "General Algebra" which seems to me to be more suitable for common use, but as I want to review today the general trend of the development of Algebra also in its timely aspect, the very often used term "Modern Algebra" may be conceded.

By suggesting the term "General Algebra" as a substitute for "Modern Algebra." I have already expressed the opinion that the characteristic of the Algebra of the present time is its huge generalisation and it will be the main purpose of this address to show the reasons which have led to investigations of such a generality as are made in Algebra now-a-days. It is an often-heard allegation that easy generalisations of well-known theorems are sometimes made with the sole purpose of publishing one more mathematical paper, and I dare not say that this charge is completely unjustified. It is not my intention to discuss the statistical density of the more and the less valuable papers on the various branches of mathematics and to increase thereby the number of the latter class. Looking at the totality of the work done by generalisation in Algebra, we must admit that our knowledge of mathematics has been considerably increased by it and that our understanding of mathematical theories has been enormously deepened. There are, of course, different degrees of understanding in Mathematics. If one has, *e.g.*, studied a very intricate proof of some theorem, followed the thread of the ideas, checked all its syllogisms, so that eventually one cannot help admitting that the theorem must be true, then one has understood the topic, but it is the lowest degree of understanding only. It can be improved by finding out the places where assumptions have been made, but this operation does not yet secure that those suppositions are really necessary. A higher degree of understanding is reached when not only the sufficient conditions for the validity of a theorem but also the necessary ones are established; even then we might not be satisfied. We want to get a full insight into the working of the mechanism of the proof. We want to know whether all the parts of that mechanism are necessary and why they are so. Moreover, we shall attempt to consider that particular theorem and its proof not as an isolated item, but as a point in a web of mathematical ideas. When we have laid bare the threads and conceived the whole of the fabric, then only we have achieved the highest degree of understanding. But on the way to this summit many new questions have arisen ; the old problem will appear to be a special case of a much more general one and this problem again requires full clarification.—Perhaps somebody in the audience may take exception to the use of terms like mechanism, web and fabric in a description of the way in which modern algebra works by generalisation ; it might be objected that mathematics develops rather like a living organism. If we compare the growth of Algebra with that of a tree, we may state that in the 20th century, Algebra has been sinking its roots deeper and deeper ; by doing so, it has not only strengthened the stem, but plenty of sap has reached the top and has produced leaves, flowers and fruits as never before. On the other hand, the 19th century appears to be a period when interest was concentrated on the upper portion of that tree and the gardeners did not further the growth of the roots more than it was strictly necessary for the development of the top. Nevertheless, what has to be compared with a sinking of roots has already started in that period, and though the end of the last century appears to be a turning point in the trend of Algebra, one has to refer to the preceding period when describing how modern Algebra has developed. All its foundations have been laid at that period already and it is interesting to state that many of the impulses leading to the modern algebraic movement came from outside Algebra.

---

* Presidential address delivered at the annual special meeting of the Calcutta Mathematical Society, held on January 2, 1943.

The notions of 'modul,' 'ring' and 'ideal' originate from Arithmetic, but a long way leads from modul as basis of a congruence of numbers used in elementary arithmetic to modul of general algebra which is synonymous with an additive abelian group. The notion of ring has been introduced by Hilbert as a set of numbers which is reproduced by addition, subtraction and multiplication, for the purpose of investigations on algebraic numbers. Now-a-days one might consider rings of polynomials and rings of matrices as the most important instances of rings. Similarly, the theory of algebraic numbers gave origin to the notion of ideal. Kummer realised that in certain rings of integral algebraic numbers, in which the factorisation is not unique, a unique factorisation can be enforced by the introduction of ideal factors. The method of ideal elements is often used in mathematics where general rules show exceptions. The most obvious instances are the points at infinity which are introduced to avoid the distinction between coplanar and intersecting pairs of straight-lines. Similarly, complex numbers have been introduced to allot roots to every algebraic equation, irrational numbers to make every pair of segments commensurable, fractional numbers to remove the distinction between ratio and quotient, and negative numbers to make subtraction of numbers possible in every case. This method of ideal element was applied by Kummer to describe the multiplicative arithmetic in rings of integral numbers by the means of elementary multiplicative arithmetic. Dedekind conceived them as certain aggregates of numbers calling these briefly "ideals." These ideals have been generalised in modern Algebra ; every homomorphism of a ring maps a particular subset of it on the unit element of the ring, and such a subset is called an ideal of the ring. Thus the notion of ideal, which was originally meant as an absolute notion, has been transformed into a relative one. This is characteristic of the way by which generalisation in modern Algebra proceeds ; another characteristicum is the connection with the operation of homomorphism. The notions homomorphism, isomorphism, automorphism, endomorphism are perhaps the most fundamental ones in the Algebra of the present time. Historically, they originate from the permutations of a finite number of objects and from the linear transformations ; both came into the picture early in the 19th century. The consideration of permutations is a prerequisite of Galois' theory, whereas the linear transformations are the algebraic equivalent of such geometrical operations as affinity, projectivity, transformation of co-ordinates, etc. Permutations, as well as linear transformations, are special cases of automorphisms and, therefore, they lead nearly automatically to the concept of 'group.' The general importance of group is derived from the fact that wherever an entity admits automorphisms these are forming a group and that every group can be considered as a group of automorphisms. Group-theory is necessary because automorphisms must be studied.—A few words to explain the great interest which is due to automorphism.

In the English language "Mathematics" has always been a plural, but in the second half of the 19th century, Mathematics became actually a plural—at least so far as the *object* of investigations is concerned. Whereas in earlier times, the object of all the mathematical investigations consisted of the couple, number and space, and these were treated with a diversity of different methods, we possess now-a-days a store of more or less well interconnected methods which are applied to the most different types of entities. To each entity belongs a particular "mathematic" (the ungrammatical singular may be excused!), *e.g.*, there is a "Geometry on the projective complex plane," an "Algebra in the ring of the matrices of order n over the ring I [x]," an "Arithmetic in the ring I of the integral numbers," etc. The number of these different Mathematics is infinite, but one can distinguish two essentially different kinds of them. In the Arithmetic of I, every element—*i.e.*, every integral number—can be determined uniquely by methods belonging to that arithmetic. In other theories it is different ; the system admits automorphisms by which it is mapped on itself, all the relations between the elements remaining invariant, thus not every element can be singled out by operations belonging to the theory itself. In Euclidean geometry, *e.g.*, all the points are equivalent, but not all the triplets of points. Between different mathematical systems there might be a (1,1) correspondence which reproduces all the operations in them. Such systems are called isomorphic, and as the entities are not considered to be as interesting as the system of laws which hold in each of them and which interconnect its elements, isomorphic systems are often considered to be equivalent or even identical. The group of automorphisms is the first invariant of a set of isomorphic systems ; to each of its sub-groups there correspond derived entities which remain invariant for that group, and conversely. So group-theory furnishes the natural classification of the derived mathematical entities in any system in which exist non-trivial automorphisms.

On the other hand, a group can be considered as a system which is the object of "a mathematic." If one omits the abstract sets, the groups and similar formations (like semi-groups, groupoids) are even the most simple systems, since there is one kind of relation only combining different elements. For this reason, the notion of group is of a generality which very often makes restrictions desirable. It has been proved to be very helpful to distinguish a particular semi-group of endomorphisms in a group; by this way one obtains "groups with operators." If one identifies these operators with the non-singular elements of the group, one comes to systems with double composition, of which the rings and the fields are the most important instances.

The notion of field originates from the notion of "fields of algebraic numbers"; the introduction of the latter was necessitated by Galois' theory; thus it comes from Algebra and not from Arithmetic. However, it owes much to the algebraic number theory, although the basis of this theory does not consist of the fields but of certain rings, of which these fields are the quotient fields. The general notion of field was given first by Heinrich Weber, but the importance of this generalisation was recognised much later only. It is characteristic of the development of Mathematics and of Algebra in particular, during the last 150 years, that the most important notions have been formed gradually, and even when they were definitely coined, they were not immediately put in the place due to them.

In this connection I may mention the notion of "indeterminate," which plays such an important role in modern algebra. It is a very easy thing to state: "*Here x is not a variable, but an object of its own kind, created by our own mind and connected with other entities by certain relations satisfying certain conditions.*" A very long development, however, was necessary before mathematicians dared to undertake such a step and to become fully conscious of what they really did. One could attribute the introduction of the indeterminate to Kronecker. He used it in order to give a basis to the notion of root of algebraic equation, which is independent of any irrationality. It was a philosophical argument which led to this creation. Kronecker objected to the classical theory of irrationality for reasons similar to those for which the Intuitionists object to it in more modern times. A popular formulation of this opinion is that only the natural numbers are made by God and that all the other notions are human artifices. So he was the first to perform the most daring human artifice, to *create* an object of his own mind, which is nothing else than just an object with a name—say x. Kronecker's method can be traced back to Cauchy who has introduced the complex number in the very same way—though the importance of that step was recognised but slowly. One of its consequences was that Algebraic Geometry got rigorous foundations and many dubious methods in this branch of Mathematics were either confirmed or swept away.

These few remarks may suffice to show that the development of modern algebra is due to reasons of two different kinds. There are "inner mathematical reasons," so to speak, to meet the needs of particular branches of Mathematics; some of them have been mentioned here. But these reasons could not have led to this particular kind of Algebra unless the general trend of mind was prepared for it.

From the beginning of the Mathematical Renaissance which started with the discovery of Differential and Integral Calculus, we are being drowned by an abundance of mathematical results, which need a careful arrangement for becoming accessible to our mind. In the first two centuries, all critical investigations on the fundamentals were—sometimes wilfully—neglected; it was a time of uncritical "going ahead." At the beginning of the 19th century a correction of the situation could not be avoided any longer. The revision of the fundamentals started from two points simultaneously, the fundamentals of Analysis and those of Geometry were investigated and fortified. The first class of investigations created the movement for rebirth of logical rigour in mathematics, whereas the second class presented us with a renewal and amplification of the *axiomatic method*. A movement which showed so much success at an early stage was bound to continue. The mathematicians were not satisfied by earning new results; they wanted to come to a deeper understanding of mathematics, to the true meaning of notions and of theorems, to the interconnecting links between them, to the limits of the range of methods, and they asked for principles which could bring order into the amazing riches of results. The longer critical investigations on the fundamentals were continued, the more the mathematical objects faded away and the interest shifted to the interconnecting relations. By considering these relations, the mathematicians obtained such ordering principles as transformation, homomorphism, group, which they needed urgently. The critical state of mind proved to be helpful to mathematical research. In Mathematics, there is no antagonism between critical investigation and productive work.

Under these conditions, Modern Algebra was being built. Let us have a glance at it now. On the basis of Group theory is erected a 3-storied building containing theories of fields, commutative rings and non-commutative rings. Galois' theory is contained in the theory of fields, whereas topics like ideals of polynomials, algebraic manifolds, theory of resultants (elimination) and integral algebraic elements (algebraic numbers) belong to the theory of commutative rings. In the floor of the non-commutative rings, the hypercomplex systems (also called 'Algebras') occupy the most conspicuous place. Close to this building is situated the theory of matrices containing the matrices over fields and those over rings, the whole of linear algebra and of quadratic forms; there are many interconnections with the subjects mentioned earlier as show the notations: matrix representation of groups and of hypercomplex systems. Tensor algebra belongs also to the matrix theory; it furnishes a connection with the theory of covariants and invariants which has been somewhat neglected in this century. This General Algebra is not unconnected with the Classical Algebra which can be defined as the Algebra over the field of the real or the field of the complex numbers; the theories of the "formally real fields" and of the "closed fields" have established some contact already. It is not possible to enumerate here all the contacts which exist between Algebra and the other principal branches of Mathematics —Arithmetic, Geometry and Analysis. The Algebra of our present time is far from being completed; it is growing rapidly, and the sketch given here may be out of date within a short time.

When reviewing the achievements of Mathematics in the last few decades and in particular those of Algebra, one cannot possibly feel like in a world where materialism and greed have suffocated idealism and all traces of independent thinking. When the world is really as bad as some moralists pretend and a great portion of the general public is willing to believe, how is it possible that generations of mathematicians are working with unabated idealism and do not cease to produce novel and original ideas?—This remark is certainly outside the competence of a professor of mathematics, but it might be conceded to me as a cheering up welcome and New-year's greetings of the Calcutta Mathematical Society to our guests who have come to this city to attend the Indian Science Congress.

SOME INFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS

By

H. C. Gupta

(Communicated by the Secretary—Received November 2, 1942)

1. The object of this paper is to evaluate some infinite integrals involving Bessel Functions. The integrals investigated here are pretty general and their integrands contain either the product of two or more Bessel Functions or the product of a Bessel Function and two Parabolic Cylinder Functions.

2. We start with Gegenbauer's integral (Watson, 1922, §13·47, (13))

$$\int_0^\infty J_n(xt)F(t,z)t^{n-1}dt = 2^{n-1}\Gamma(n)x^{-n}G(z), \qquad \ldots \quad (2{\cdot}1)$$

where $F(t,z) = \dfrac{J_\nu\{a\sqrt{t^2+z^2}\}\,J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\lambda+\nu)}}$, and $G(z) = \dfrac{J_\nu(az)J_\lambda(az)}{z^{\lambda+\nu}}$,

provided that $x > 2a$, $R(\lambda+\nu+\frac{3}{2}) > R(n) > 0$.

Now multiply both sides of (2.1) by $x^{2l+n-1}e^{-\frac{1}{4}x^2}W_{k,m}(\frac{1}{2}x^2)$ and integrate with respect to x from 0 to ∞. Then

$$\int_0^\infty x^{2l+n-1}e^{-\frac{1}{4}x^2}W_{k,m}(\tfrac{1}{2}x^2)dx\int_0^\infty J_n(xt)F(t,z)t^{n-1}dt$$

$$= 2^{n-1}\Gamma(n)G(z)\int_0^\infty x^{2l-1}e^{-\frac{1}{4}x^2}W_{k,m}(\tfrac{1}{2}x^2)dx \qquad \ldots \quad (2{\cdot}2)$$

$$= \frac{2^{n+l-2}\Gamma(n)\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}G(z)$$

on using a formula given by Goldstein (1932).

The left-hand side of (2·2), on changing the order of integration, becomes

$$\int_0^\infty F(t,z)t^{n-1}dt\int_0^\infty x^{2l+n-1}e^{-\frac{1}{4}x^2}W_{k,m}(\tfrac{1}{2}x^2)J_n(xt)dx.$$

Evaluating the inner integral of this by a known formula (Varma, 1939), we get

$$\int_0^\infty \frac{t^{2n-1}J_\nu\{a\sqrt{t^2+z^2}\}J_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\lambda+\nu)}}$$

$$\times\, {}_2F_2\left(\begin{matrix} l+m+n+\frac{1}{2},\ l+n-m+\frac{1}{2} \\ n+1,\ l+n-k+1 \end{matrix} - \tfrac{1}{2}t^2\right)dt$$

$$= \frac{2^{n-1}\Gamma(n)\Gamma(n+1)\Gamma(l+n-k+1)\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)\Gamma(l+m+n+\frac{1}{2})\Gamma(l+n-m+\frac{1}{2})z^{\lambda+\nu}}\; J_\nu(az)\,J_\lambda(az) \quad \ldots \quad (2{\cdot}3)$$

The change in the order of integration effected in (2·2) is justified if the double integral

$$\int_0^\infty |t^{n-1}\mathrm{F}(t, z)|\,dt \int_0^\infty |x^{2l+n-1}e^{-\frac{1}{4}x^2}\mathrm{W}_{k,\,m}(\tfrac{1}{2}x^2)\mathrm{J}_n(xt)|\,dx$$

is convergent, which is so if $\mathrm{R}(\lambda+\nu+\frac{5}{2}) > \mathrm{R}(n) > 0$ and $\mathrm{R}(n+l\pm m) > -\frac{1}{2}$. Hence the result (2·3) is true when $\mathrm{R}(l\pm m+n) > -\frac{1}{2}$ and $\mathrm{R}(\lambda+\nu+\frac{5}{2}) > \mathrm{R}(n) > 0$.

3. We shall now discuss some interesting particular cases of (2·3). The generalised hypergeometric function ${}_2\mathrm{F}_2$ degenerates into ${}_1\mathrm{F}_1$, when (i) $l+m=\frac{1}{2}$, or (ii) $m=k-\frac{1}{2}$, or (iii) $l-m=\frac{1}{2}$, or (iv) $m+k=\frac{1}{2}$.

Cases (i) and (iii) give the integral

$$\int_0^\infty \frac{t^{2n-1}\mathrm{J}_\nu\{a\sqrt{t^2+z^2}\}\mathrm{J}_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\lambda+\nu)}}\cdot{}_1\mathrm{F}_1\left(\begin{matrix}2l+n\\ l+n-k+1\end{matrix}\; -\tfrac{1}{2}t^2\right)dt$$

$$= \frac{2^{n-1}\Gamma(n)\Gamma(l+n-k+1)\Gamma(2l)\mathrm{J}_\nu(az)\mathrm{J}_\lambda(az)}{\Gamma(l-k+1)\Gamma(2l+n)z^{\lambda+\nu}},\ \mathrm{R}(\lambda+\nu+4l+2)>0,\ \mathrm{R}(n)>0.$$

Cases (ii) and (iv) give the integral

$$\int_0^\infty \frac{t^{2n-1}\mathrm{J}_\nu\{a\sqrt{t^2+z^2}\}\mathrm{J}_\lambda\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\frac{1}{2}(\lambda+\nu)}}\cdot{}_1\mathrm{F}_1(q+n;\ n+1;\ -\tfrac{1}{2}t^2)dt$$

$$= \frac{2^{n-1}\Gamma(n+1)\Gamma(n)\Gamma(q)\mathrm{J}_\nu(az)\mathrm{J}_\lambda(az)}{\Gamma(q+n)z^{\lambda+\nu}},\ \ \mathrm{R}(\lambda+\nu+2q+2)>0,\ \mathrm{R}(n)>0,$$

which, as can easily be seen, on putting $2q=l-n+1$, and using the duplication formula for Gamma function, reduces to a result recently given by Miss Sinha (1942).

4. We know that if $\phi_1(p) \doteqdot f_1(x)$, *i.e.*,

$$\phi_1(p) = p\int_0^\infty e^{-px}f_1(x)dx,$$

then
$$\phi_1(p/t) \doteqdot f_1(xt),$$
where t is a constant > 0.

Applying this theorem to Van der Pol's result (1929), *viz.*,

$$\frac{p}{(p^2+1)^{n+\frac{1}{2}}} \doteqdot \frac{\sqrt{\pi}}{2^n\Gamma(n+\frac{1}{2})}x^n\mathrm{J}_n(x),\ \ n>-\tfrac{1}{2},$$

we get
$$\frac{2^n\Gamma(n+\frac{1}{2})(p/t)}{\sqrt{\pi}(1+p^2/t^2)^{n+\frac{1}{2}}} \doteqdot (xt)^n\mathrm{J}_n(xt) \qquad \ldots \quad (4·1)$$

Now a form of Parseval's theorem recently given by Goldstein (1932) is that if $\phi_1(p)$ and $\phi_2(p)$ are operationally related to $f_1(x)$ and $f_2(x)$ respectively, then

$$\int_0^\infty \frac{\phi_1(x)f_2(x)}{x}\,dx = \int_0^\infty \frac{\phi_2(x)f_1(x)}{x}\,dx.$$

Applying this theorem to (4·1) and the following result due to Meijer (1937)

$$\pi^{-\frac{1}{2}}\Gamma(m+\tfrac{1}{2})p\mathrm{D}_{-m-\frac{1}{2}}(pe^{\frac{1}{4}\pi i})\,\mathrm{D}_{-m-\frac{1}{2}}(pe^{-\frac{1}{4}\pi i}) \doteqdot \mathrm{J}_m(\tfrac{1}{2}x^2), \qquad \ldots \quad (4{\cdot}2)$$

$$\mathrm{R}(m) > -\tfrac{1}{2}$$

we have

$$\frac{\Gamma(m+\frac{1}{2})}{\sqrt{\pi}}\int_0^\infty \mathrm{D}_{-m-\frac{1}{2}}(xe^{\frac{1}{4}\pi i})\mathrm{D}_{-m-\frac{1}{2}}(xe^{-\frac{1}{4}\pi i})(xt)^n\mathrm{J}_n(xt)dx$$

$$= \frac{2^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}\int_0^\infty \frac{t^{2n}\,\mathrm{J}_m(\frac{1}{2}x^2)}{(x^2+t^2)^{n+\frac{1}{2}}}\,dx.$$

Evaluating the last integral (Watson, 1922, p. 436), we get

$$\int_0^\infty x^n\mathrm{J}_n(xt)\,\mathrm{D}_{-m-\frac{1}{2}}(xe^{\frac{1}{4}\pi i})\,\mathrm{D}_{-m-\frac{1}{2}}(xe^{-\frac{1}{4}\pi i})\,dx$$

$$= \frac{\pi 2^{n-1}t^{-n}}{\Gamma(m+\frac{1}{2})\sin(m-n+1)\pi}\left[\sum_{r=0}^{\infty}\frac{(-)^r\Gamma(\frac{1}{2}+m+2r)\,(\frac{1}{4}t^2)^{m+2r}}{r!\,\Gamma(m+r+1)\Gamma(m-n+2r+1)}\right.$$

$$\left.-\sum_{r=0}^{\infty}\frac{\pi(\frac{1}{4}t^2)^{n+r}\Gamma(n+r+\frac{1}{2})}{r!\,\Gamma\{\frac{1}{2}(m-n-r+1)\}\Gamma\{\frac{1}{2}(n-m+r+1)\}\Gamma\{\frac{1}{2}(n+m+r+2)\}\Gamma\{\frac{1}{2}(n-m+r+2)\}}\right]$$

$$= \frac{\Gamma(n-m)\,(\frac{1}{2}t)^{2m-n}}{2\Gamma(m+1)}\cdot{}_2\mathrm{F}_3\left(\begin{matrix}\frac{1}{2}m+\frac{1}{4},\ \frac{1}{2}m+\frac{3}{4}\ ; \\ m+1,\ \frac{1}{2}(m-n+1),\ \frac{1}{2}(m-n+2)\ ;\end{matrix}\ -\tfrac{1}{16}t^4\right)$$

$$+\frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2}m-\frac{1}{2}n)(\frac{1}{2}t)^n}{4\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{2}m+\frac{1}{2}n+1)}\cdot{}_2\mathrm{F}_3\left(\begin{matrix}\frac{1}{2}n+\frac{1}{4},\ \frac{1}{2}n+\frac{3}{4}\ ; \\ \frac{n+m+2}{2},\ \frac{n-m+2}{2},\ \frac{1}{2}\ ;\end{matrix}\ -\tfrac{1}{16}t^4\right)$$

$$+\frac{\Gamma(n+\frac{3}{2})\Gamma\left(\frac{m-n-1}{2}\right)(\frac{1}{2}t)^{n+2}}{4\Gamma(m+\frac{1}{2})\Gamma\left(\frac{n+m+3}{2}\right)}\cdot{}_2\mathrm{F}_3\left(\begin{matrix}\frac{1}{2}n+\frac{3}{4},\ \frac{1}{2}n+\frac{5}{4}\ ; \\ \frac{3}{2},\ \frac{m+n+3}{2},\ \frac{n-m+3}{2}\ ;\end{matrix}\ -\tfrac{1}{16}t^4\right)(4{\cdot}3)$$

provided that $t > 0$ and $\mathrm{R}(2m+2) > \mathrm{R}(n+\frac{1}{2}) > 0$, $\mathrm{R}(m) > -\frac{1}{2}$.

5. We now establish a lemma, a particular case of which is required in evaluating an integral.

Lemma.

$$\int_0^\infty x^\nu\,\mathrm{D}_{-m-\frac{1}{2}}(xe^{\frac{1}{4}\pi i})\,\mathrm{D}_{-m-\frac{1}{2}}(xe^{-\frac{1}{4}\pi i})\,dx$$

$$= \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}m-\frac{1}{4}\nu)\Gamma(\nu+1)}{2^{\nu+2}\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{4}\nu+\frac{1}{2}m+1)}, \quad \mathrm{R}(2m) > \mathrm{R}(\nu) > -1.$$

To prove this we apply the theorem of §4 to the operational relations (4·2) and (Humbert, 1934)

$$\frac{1}{p^\nu} \doteqdot \frac{x^\nu}{\Gamma(\nu+1)}, \quad \mathrm{R}(\nu) > -1.$$

We then have

$$\frac{\Gamma(m+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu+1)}\int_0^\infty x^\nu \, D_{-m-\frac{1}{2}}(xe^{\frac{1}{4}\pi i})\, D_{-m-\frac{1}{2}}(xe^{-\frac{1}{4}\pi i})\,dx$$

$$=\int_0^\infty \frac{J_m(\frac{1}{4}x^2)}{x^{\nu+1}}dx = \frac{\Gamma(\frac{1}{2}m-\frac{1}{4}\nu)}{2^{\nu+2}\Gamma(\frac{1}{4}\nu+\frac{1}{2}m+1)}, \quad R(2m)>R(\nu)\geq -1$$

on using the formula (Watson, 1922, p. 391)

$$\int_0^\infty \frac{J_\nu(t)\,dt}{t^{\nu-\mu+1}} = \frac{\Gamma(\frac{1}{2}\mu)}{2^{\nu-\mu+1}\,\Gamma(\nu-\frac{1}{2}\mu+1)}, \; 0 \leq R(\mu) < R(\nu)+\tfrac{3}{2}.$$

This proves the lemma.

6. Now multiply both sides of (2·1) by

$$x^n D_{-m-\frac{1}{2}}(xe^{\frac{1}{4}\pi i})\, D_{-m-\frac{1}{2}}(xe^{-\frac{1}{4}\pi i})$$

and integrate with respect to x from 0 to ∞. Then

$$\int_0^\infty x^n D_{-m-\frac{1}{2}}(xe^{\frac{1}{4}\pi i}) D_{-m-\frac{1}{2}}(xe^{-\frac{1}{4}\pi i})dx \int_0^\infty J_n(xt)t^{n-1}F(t,s)dt$$

$$= 2^{n-1}\Gamma(n)G(s) \int_0^\infty D_{-m-\frac{1}{2}}(xe^{\frac{1}{4}\pi i}) D_{-m-\frac{1}{2}}(xe^{-\frac{1}{4}\pi i})dx$$

Changing the order of integration [valid when $R(\lambda+\nu+\frac{3}{2}) > R(n) > 0$ and $R(2m+\frac{3}{2}) > R(n)$] in the double integral and using (4·8) and the above lemma, we have

$$\int_0^\infty \frac{J_\nu\{a\sqrt{t^2+s^2}\}J_\lambda\{a\sqrt{t^2+s^2}\}}{(t^2+s^2)^{\frac{1}{2}(\lambda+\nu)}}\phi(t)dt = \frac{\sqrt{\pi}\,2^{n-2}\Gamma(n)J_\nu(as)\,J_\lambda(as)}{m\Gamma(m+\frac{1}{2})s^{\lambda+\nu}},$$

where $\phi(t)$ is the expression on the right side of (4·8) provided that

$$R(\lambda+\nu+2) \geq 2m \geq 0 \text{ and } R(\lambda+\nu) \geq 2n \geq 0.$$

7. Lastly, multiply both sides of (2·1) by $x^{-1}exp(-1/x)$ and integrate with respect to x from 0 to ∞, and proceed in the above manner. We get on using the integrals (Hardy, 1926)

$$\int_0^\infty J_n(s)s^{-1}exp(-2k/s)ds = 2J_n(2\sqrt{k})K_n(2\sqrt{k})$$

and

$$\int_0^\infty exp(-1/x)\,x^{-m}dx = \Gamma(m-1), \; R(m) > 1,$$

that

$$\int_0^\infty t^{n-1}\frac{J_\nu\{a\sqrt{t^2+s^2}\}J_\lambda\{a\sqrt{t^2+s^2}\}}{(t^2+s^2)^{\frac{1}{2}(\lambda+\nu)}}\,J_n(\sqrt{2t})K_n(\sqrt{2t})dt$$

$$= 2^{n-2}\Gamma^2(n)\,\frac{J_\nu(as)\,J_\lambda(as)}{s^{\lambda+\nu}}, \qquad \text{provided that } R(n) > 0.$$

In conclusion, I wish to express my gratitude to Dr. R. S. Varma for suggesting the problem and for taking a keen interest in the preparation of this paper.

CHRIST CHURCH COLLEGE,
CAWNPORE.

References

Goldstein, S., 1932, *Proc. Lond. Math. Soc.* (2), **34**, 106.
Hardy, G. H., 1926, *Messenger of Mathematics*, **56**, 190.
Humbert, P., 1934, *Le Calcul Symbolque*, 27-28.
Meijer, C. S., 1935, *Phil. Mag.* (7), **24**, 107.
Pol, B. Van der, 1929, *Phil Mag.* (7), **8**, 882.
Sinha, S. (Miss), 1942, *Bull. Cal. Math. Soc.*, **34**, 72.
Varma, R. S. 1939, *Jour. Ind. Math. Soc.* (8), **1**, 29.
Watson, G. N., 1922, *Theory of Bessel Functions* (Cambridge).

STRESSES DUE TO FORCES AND COUPLES ACTING IN THE INTERIOR OF AN INFINITE ELASTIC SLAB PLACED ON A RIGID FOUNDATION

By

Bibhutibhusan Sen

(Received January 20, 1948)

Introduction

Stresses due to isolated *forces, centres of compression* and *centres of rotation* operating in the interior of a *semi-infinite* elastic solid have been determined in a previous paper[*] (Sen, 1940). The object of the present one is to find the stresses and displacements due to a force or a couple acting *inside* an infinite elastic *slab* of finite and uniform thickness which rests on and adheres to a perfectly rigid foundation. These problems seem to have important applications in soil mechanics; for we may suppose the infinite slab considered here to be formed by an infinite layer of soft ground resting on a rigid foundation. Similar problems of forces and couples acting on the *surface* of such a slab have already been discussed in a different manner by Passer (1935) and the author (Sen, 1937) respectively. The method followed in this paper is the same as that of Paper I (Sen, 1940) of which some results are assumed here without proof. In the solutions the stresses have been expressed as integrals. No attempt has been made to have numerical values of the stresses from these integrals in the case of a force acting inside the slab. But for the case of a couple, a method of finding approximately the stresses on the plane base is given.

1. *Method of solution*

We suppose that the upper surface of the slab is given by the plane $z=0$, the axis of z being drawn into the body.

We can represent the other face which rests on a rigid foundation by the plane $z=k$. In Paper I, we have seen how the stresses $(\widehat{xx}, \widehat{xy}, \widehat{xz}, \widehat{yy}, \widehat{yz}, \widehat{zz})$ and displacements (u, v, w) due to a force or some nucleus of strain operating at the point $(0, 0, c)$ inside a *semi-infinite* solid can be found. These stresses and displacements vanish at infinite distances from the plane surface $z=0$ which is free from stress. The above-mentioned stresses and displacements being known, we are now to find another system of stresses which still keep the plane surface $z=0$ free from stress and yield displacements nullifying (u, v, w) on the surface $z=k$. Let us use the subscript 3 for the new system.

---

[*] This paper will be referred to as Paper I. —There is a misprint in the result (2.4) of the paper where the expression for V_2 should be log $(z+c+R_2)$ and not log $(z+R_2)$.

Then the precise statement of the conditions is that a system of stresses is to be found which will give

$$\widehat{xz}_3 = \widehat{yz}_3 = \widehat{zz}_3 = 0, \qquad \text{when } z = 0, \qquad \ldots \quad (1.1)$$

while the displacements (u_3, v_3, w_3) producing these stresses should be such that

$$u + u_3 = v + v_3 = w + w_3 = 0, \qquad \text{when } z = k, \qquad \ldots \quad (1.2)$$

(u, v, w) being the displacements obtained for a force or a couple (as the case may be) operating at the point $(0, 0, c)$ inside a semi-infinite solid bounded by the plane $z = 0$. The resultant displacements at any point of the slab will be

$$u + u_3,\ v + v_3,\ w + w_3, \qquad \ldots \quad (1.3)$$

and the resultant stresses at the point will be given by

$$\left.\begin{array}{l} \widehat{xx} + \widehat{xx}_3,\ \widehat{xy} + \widehat{xy}_3,\ \widehat{xz} + \widehat{xz}_3, \\ \widehat{yy} + \widehat{yy}_3,\ \widehat{yz} + \widehat{yz}_3,\ \widehat{zz} + \widehat{zz}_3. \end{array}\right\} \qquad \ldots \quad (1.4)$$

The new system of stresses must be physically possible in an elastic solid and consequently the consistency equations and the equations of equilibrium must be satisfied by them. Hence we shall have [*vide* Paper I (1.4)]

$$\left.\begin{aligned} \widehat{xz}_3 &= -\frac{1}{2(1+\sigma)}\, z \frac{\partial \Theta_3}{\partial x} + \psi_1, \\ \widehat{yz}_3 &= -\frac{1}{2(1+\sigma)}\, z \frac{\partial \Theta_3}{\partial y} + \psi_2, \\ \widehat{zz}_3 &= -\frac{1}{2(1+\sigma)}\, z \frac{\partial \Theta_3}{\partial z} + \psi_3, \end{aligned}\right\} \qquad \ldots \quad (1.5)$$

where σ is Poisson's ratio, $\Theta_3 = \widehat{xx}_3 + \widehat{yy}_3 + \widehat{zz}_3$ and ψ_1, ψ_2 and ψ_3 are harmonic functions vanishing on the plane $z = 0$.

These stresses expressed in the above form ensure the satisfaction of the conditions (1.1). Usually, some constant multipliers [to be determined from the conditions (1.2)] will be associated with ψ_1, ψ_2 and ψ_3.

Moreover, we have from the equations of equilibrium,

$$\frac{\partial \Theta_3}{\partial z} = 2(1+\sigma)\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right), \qquad \ldots \quad (1.6)$$

and from stress-strain relations,

$$\left.\begin{aligned} \frac{\partial w_3}{\partial z} &= \frac{1}{E}\left[(1+\sigma)\widehat{zz}_3 - \sigma \Theta_3\right], \\ \frac{\partial v_3}{\partial z} + \frac{\partial w_3}{\partial y} &= \frac{2(1+\sigma)}{E}\,\widehat{yz}_3, \\ \frac{\partial u_3}{\partial z} + \frac{\partial w_3}{\partial x} &= \frac{2(1+\sigma)}{E}\,\widehat{xz}_3. \end{aligned}\right\} \qquad \ldots \quad (1.7)$$

We can find expressions for u_3, v_3 and w_3 (involving the constant multipliers of ψ_1, ψ_2 and ψ_3) which satisfy the above relations. Since (u, v, w) are known then from the conditions (1.2), the unknown constants can be found. Thus the new system of displacements and hence the stresses are completely determined.

2. *A force operating at the point (0, 0, c) along the z axis*

If a force of magnitude P act along the z-axis at the point $(0, 0, c)$ of a *semi-infinite* elastic solid, with the surface $z=0$ free from stress, we have the displacements [Paper I, (2.9)]

$$\left.\begin{aligned}
u &= \frac{P(1+\sigma)x}{8\pi(1-\sigma)E}\left[\frac{z-c}{R_1^3}+\frac{(3-4\sigma)(z-c)}{R_2^3}-\frac{4(1-\sigma)(1-2\sigma)}{R_2(R_2+z+c)}+\frac{6cz(z+c)}{R_2^5}\right],\\
v &= \frac{P(1+\sigma)y}{8\pi(1-\sigma)E}\left[\frac{z-c}{R_1^3}+\frac{(3-4\sigma)(z-c)}{R_2^3}-\frac{4(1-\sigma)(1-2\sigma)}{R_2(R_2+z+c)}+\frac{6cz(z+c)}{R_2^5}\right],\\
w &= \frac{P(1+\sigma)}{8\pi(1-\sigma)E}\left[\frac{3-4\sigma}{R_1}+\frac{(z-c)^2}{R_1^3}+\frac{5-12\sigma+8\sigma^2}{R_2}+\frac{(3-4\sigma)(z+c)^2-2cz}{R_2^3}\right.\\
&\qquad\left.+\frac{6cz(z+c)^2}{R_2^5}\right],
\end{aligned}\right\} \quad (2.1)$$

where

$$R_1^2=x^2+y^2+(z-c)^2, \text{ and } R_2^2=x^2+y^2+(z+c)^2. \quad \ldots \quad (2.2)$$

Writing r for $\sqrt{x^2+y^2}$, we have

$$\left.\begin{aligned}
\frac{1}{R_1} &= \int_0^\infty e^{-\alpha(z-c)}J_0(\alpha r)d\alpha \quad (z>c),\\
\frac{1}{R_2} &= \int_0^\infty e^{-\alpha(z+c)}J_0(\alpha r)d\alpha,
\end{aligned}\right\} \quad \ldots \quad (2.3)$$

$J_0(\alpha r)$ being a Bessel's function of the first kind and the zeroeth order. Substituting these values in (2.1) we shall have, on the surface $z=k$,

$$\left.\begin{aligned}
[u]_{z=k} &= -\frac{1+\sigma}{E}\int_0^\infty L(\alpha)\ \frac{\partial}{\partial x}[J_0(\alpha r)]d\alpha,\\
[v]_{z=k} &= -\frac{1+\sigma}{E}\int_0^\infty L(\alpha)\ \frac{\partial}{\partial y}[J_0(\alpha r)]d\alpha,\\
[w]_{z=k} &= \frac{1+\sigma}{E}\int_0^\infty M(\alpha)J_0(\alpha r)d\alpha,
\end{aligned}\right\} \quad \ldots \quad (2.4)$$

where

$$\left.\begin{aligned} L(\alpha) &= \frac{P}{8\pi(1-\sigma)}\Big[(k-c)e^{-\alpha(k-c)} + (3-4\sigma)(k-c)e^{-\alpha(k+c)} \\ &\qquad - \frac{4(1-\sigma)(1-2\sigma)}{\alpha}e^{-\alpha(k+c)} + 2\alpha cke^{-\alpha(k+c)}\Big], \\ M(\alpha) &= \frac{P}{8\pi(1-\sigma)}\Big[(3-4\sigma)e^{-\alpha(k-c)} + (k-c)\alpha e^{-\alpha(k-c)} \\ &\qquad + (5-12\sigma+8\sigma^2)e^{-\alpha(k+c)} + (3-4\sigma)(k+c)\alpha e^{-\alpha(k+c)} \\ &\qquad + 2\alpha^2 cke^{-\alpha(k+c)}\Big], \end{aligned}\right\} \quad \dots \quad (2.5)$$

To find the compensating displacements u_3, v_3, w_3, let us put (1.5) as

$$\left.\begin{aligned} \widehat{xz}_3 &= -\frac{1}{2(1+\sigma)}z\frac{\partial \odot_3}{\partial x} + \int_0^\infty A(\alpha)\sinh \alpha z \frac{\partial}{\partial x}[J_0(\alpha r)]d\alpha, \\ \widehat{yz}_3 &= -\frac{1}{2(1+\sigma)}z\frac{\partial \odot_3}{\partial y} + \int_0^\infty A(\alpha)\sinh \alpha z \frac{\partial}{\partial y}[J_0(\alpha r)]d\alpha, \\ \widehat{zz}_3 &= -\frac{1}{2(1+\sigma)}z\frac{\partial \odot_3}{\partial z} + \int_0^\infty \alpha B(\alpha)\sinh \alpha z \, J_0(\alpha r)d\alpha, \end{aligned}\right\} \quad \dots \quad (2.6)$$

where $A(\alpha)$ and $B(\alpha)$ [which will be henceforth written simply as A and B] are some functions of α. These expressions make $\widehat{xz}_3 = \widehat{yz}_3 = \widehat{zz}_3 = 0$ on the plane $z=0$. The relation (1.6) then gives

$$\frac{\partial \odot_3}{\partial z} = 2(1+\sigma)\int_0^\infty \alpha^2[B\cosh \alpha z - A\sinh \alpha z]J_0(\alpha r)d\alpha,$$

which is satisfied, if we take

$$\odot_3 = 2(1+\sigma)\int_0^\infty \alpha[B\sinh \alpha z - A\cosh \alpha z]J_0(\alpha r)d\alpha, \quad \dots \quad (2.7)$$

This value of $\odot_3$ being substituted into (2.6), we get

$$\left.\begin{aligned} \widehat{xz}_3 &= \int_0^\infty [-\alpha z(B\sinh \alpha z - A\cosh \alpha z) + A\sinh \alpha z]\frac{\partial}{\partial x}[J_0(\alpha r)]d\alpha, \\ \widehat{yz}_3 &= \int_0^\infty [-\alpha z(B\sinh \alpha z - A\cosh \alpha z) + A\sinh \alpha z]\frac{\partial}{\partial y}[J_0(\alpha r)]d\alpha, \\ \widehat{zz}_3 &= \int_0^\infty [-\alpha^2 z(B\cosh \alpha z - A\sinh \alpha z) + \alpha B\sinh \alpha z]J_0(\alpha r)d\alpha. \end{aligned}\right\} \quad \dots \quad (2.8)$$

Particular values of u_3, v_3 and w_3 satisfying the stress-train relations (1.7) are then given by

$$\left.\begin{aligned}
u_3 &= \frac{1+\sigma}{E}\int_0^\infty \Big[\frac{A}{\alpha}\{2(1-\sigma)\cosh \alpha z+\alpha z\sinh \alpha z\} \\
&\qquad -\frac{B}{\alpha}\{(1-2\sigma)\sinh \alpha z+\alpha z\cosh \alpha z\}\Big]\frac{\partial}{\partial x}[J_0(\alpha r)]\,d\alpha, \\
v_3 &= \frac{1+\sigma}{E}\int_0^\infty \Big[\frac{A}{\alpha}\{2(1-\sigma)\cosh \alpha z+\alpha z\sinh \alpha z\} \\
&\qquad -\frac{B}{\alpha}\{(1-2\sigma)\sinh \alpha z+\alpha z\cosh \alpha z\}\Big]\frac{\partial}{\partial y}[J_0(\alpha r)]\,d\alpha, \\
w_3 &= \frac{1+\sigma}{E}\int_0^\infty [A\{\alpha z\cosh \alpha z-(1-2\sigma)\sinh \alpha z\} \\
&\qquad +B\{2(1-\sigma)\cosh \alpha z-\alpha z\sinh \alpha z\}]J_0(\alpha r)\,d\alpha.
\end{aligned}\right\} \quad \ldots \quad (2\cdot 9)$$

The conditions (1.2) will be satisfied if

$$A[2(1-\sigma)\cosh \alpha k+\alpha k\sinh \alpha k]-B[(1-2\sigma)\sinh \alpha k+\alpha k\cosh \alpha k]=\alpha L(\alpha)$$

$$A[-\alpha k\cosh \alpha k+(1-2\sigma)\sinh \alpha k]-B[2(1-\sigma)\cosh \alpha k-\alpha k\sinh \alpha k]=M(\alpha)$$

whence we get

$$\left.\begin{aligned}
A &= \frac{1}{\triangle}[-\alpha L(\alpha)\{2(1-\sigma)\cosh \alpha k-\alpha k\sinh \alpha k\} \\
&\qquad +M(\alpha)\{(1-2\sigma)\sinh \alpha k+\alpha k\cosh \alpha k\}], \\
B &= \frac{1}{\triangle}[-\alpha L(\alpha)\{(1-2\sigma)\sinh \alpha k-\alpha k\cosh \alpha k\} \\
&\qquad +M(\alpha)\{2(1-\sigma)\cosh \alpha k+\alpha k\sinh \alpha k\}], \\
\text{where}\quad \triangle &= [(1-2\sigma)^2\sinh^2\alpha k-\alpha^2k^2\cosh^2\alpha k] \\
&\qquad -[4(1-\sigma)^2\cosh^2\alpha k-\alpha^2k^2\sinh^2\alpha k].
\end{aligned}\right\}$$

Values of A and B being known, u_3, v_3, w_3 and hence the stress components given by these displacements can be found. It can be easily seen that the definite integrals involved in the expressions for u, v and w are convergent when $\alpha \to 0$ as well as when $\alpha \to \infty$. For given values of k and σ, these integrals can be evaluated by the method of quadratures.

3. *A couple operating at the point* (0, 0, *c*).

Let there be a centre of rotation about the axis of z operating at the point (0, 0, c). The forces applied to the body in the neighbourhood of the point being statically equivalent to a couple of moment Q, we have in a *semi-infinite* solid bounded by the plane $z=0$ [*vide* Paper I, §3] the stresses

$$\left.\begin{aligned} \widehat{xz} &= \frac{3Qy}{8\pi}\left[\frac{z-c}{R_1^{\,3}}+\frac{z+c}{R_2^{\,5}}\right], \\ \widehat{yz} &= -\frac{3Qx}{8\pi}\left[\frac{z-c}{R_1^{\,5}}+\frac{z+c}{R_2^{\,5}}\right], \\ \widehat{zz} &= 0, \qquad\qquad \odot=0, \end{aligned}\right\} \quad \ldots \quad (3.1)$$

and the displacements

$$\left.\begin{aligned} w &= 0, \\ u &= -\frac{Q(1+\sigma)y}{4\pi E}\left[\frac{1}{R_1^{\,3}}+\frac{1}{R_2^{\,3}}\right], \\ v &= \frac{Q(1+\sigma)x}{4\pi E}\left[\frac{1}{R_1^{\,3}}+\frac{1}{R_2^{\,3}}\right]. \end{aligned}\right\} \quad \ldots \quad (3.2)$$

To find the displacements u_3 and v_3 which nullify the above displacements on the plane $z=k$, let us put the results (1.5) as

$$\left.\begin{aligned} \widehat{xz}_3 &= -\frac{1}{2(1+\sigma)}\, z\,\frac{\partial\odot_3}{\partial x}-\int_0^\infty C(a)\sinh az\,\frac{\partial}{\partial y}\,[J_0(ar)]\,da, \\ \widehat{yz}_3 &= -\frac{1}{2(1+\sigma)}\, z\,\frac{\partial\odot_3}{\partial y}+\int_0^\infty C(a)\sinh az\,\frac{\partial}{\partial x}[J_0(ar)]\,da, \\ \widehat{zz}_3 &= 0. \end{aligned}\right\} \quad \ldots \quad (3.3)$$

These satisfy the consistency equations as well as the conditions that

$$\widehat{xz}_3=\widehat{yz}_3=\widehat{zz}_3=0 \quad \text{when } z=0.$$

The relation (1.6) is also satisfied if we take $\odot_3=0$. In that case we have

$$\left.\begin{aligned} \widehat{xz}_3 &= -\int_0^\infty C(a)\sinh az\,\frac{\partial}{\partial y}\,[J_0(ar)]\,da, \\ \widehat{yz}_3 &= +\int_0^\infty C(a)\sinh az\,\frac{\partial}{\partial x}\,[J_0(ar)]\,da. \end{aligned}\right\} \quad \ldots \quad (3.4)$$

The stress-strain relations (1.7) will also be satisfied provided we take

$$\left.\begin{aligned} w_3 &= 0, \\ u_3 &= -\frac{2(1+\sigma)}{E}\int_0^\infty \frac{C(a)}{a}\cosh az\,\frac{\partial}{\partial y}[J_0(ar)]\,da, \\ v_3 &= \frac{2(1+\sigma)}{E}+\int_0^\infty \frac{C(a)}{a}\cosh az\,\frac{\partial}{\partial x}[J_0(ar)]\,da. \end{aligned}\right\} \quad \ldots \quad (3.5)$$

Again from (3.2) we have on the plane $z=k$,

$$\left.\begin{aligned}[u]_{z=k}&=\frac{Q(1+\sigma)}{2\pi E}\int_0^\infty e^{-\alpha k}\cosh\alpha c\,\frac{\partial}{\partial y}\,[J_0(\alpha r)]\,d\alpha,\\ [v]_{z=k}&=-\frac{Q(1+\sigma)}{2\pi E}\int_0^\infty e^{-\alpha k}\cosh\alpha c\,\frac{\partial}{\partial x}[J_0(\alpha r)d\alpha.\end{aligned}\right\}\quad\ldots\quad(3.6)$$

Since $u+u_3=0$, $v+v_3=0$ on the plane $z=k$, we must have

$$C(\alpha)=\frac{Q}{4\pi}\,\alpha e^{-\alpha k}\,\frac{\cosh\alpha c}{\cosh\alpha k}\,.\quad\ldots\quad(3.7)$$

Thus we get at any point the resultant displacements

$$u+u_3,\ v+v_3,\ 0,$$

where u_3 and v_3 are given by

$$\left.\begin{aligned}u_3&=-\frac{Q(1+\sigma)}{2\pi E}\int_0^\infty e^{-\alpha k}\,\frac{\cosh\alpha c}{\cosh\alpha k}\cosh\alpha z\,\frac{\partial}{\partial y}\,[J_0(\alpha r)]d\alpha,\\ v_3&=\frac{Q(1+\sigma)}{2\pi E}\int_0^\infty e^{-\alpha k}\,\frac{\cosh\alpha c}{\cosh\alpha k}\cosh\alpha z\,\frac{\partial}{\partial y}\,[J_0(\alpha r)]d\alpha,\end{aligned}\right\}\quad\ldots\quad(3.8)$$

and the resultant stresses on a plane $z=$const. are

$$\widehat{xz}+\widehat{xz}_3,\ \widehat{yz}+\widehat{yz}_3,\ 0,$$

where

$$\left.\begin{aligned}\widehat{xz}_3&=-\frac{Q}{4\pi}\int_0^\infty \alpha e^{-\alpha k}\,\frac{\cosh\alpha c}{\cosh\alpha z}\ \sinh\alpha z\,\frac{\partial}{\partial y}\,[J_0(\alpha r)]d\alpha,\\ \widehat{yz}_3&=\frac{Q}{4\pi}\int_0^\infty \alpha e^{-\alpha k}\,\frac{\cosh\alpha c}{\cosh\alpha z}\ \sinh\alpha z\,\frac{\partial}{\partial x}\,[J_0(\alpha r)]d\alpha.\end{aligned}\right\}\quad\ldots\quad(3.9)$$

Other non-vanishing stress components can be calculated from (3.2) and (3.8). On the surface $z=k$ we have

$$\left.\begin{aligned}[\widehat{\theta z}]_{z=k}&=-\frac{Q}{4\pi}\int_0^\infty \alpha^2 e^{-\alpha k}\cosh\alpha c\tanh\alpha z\ J_1(\alpha r)d\alpha\\ &=-\frac{Q}{4\pi}\,I.\end{aligned}\right\}\quad\ldots\quad(3.10)$$

4. *Approximations*

In this section a method of finding the approximate values of the integral I given in (3.10) will be discussed. In the first instance we shall assume that k is small in comparison with r, the distance of the point considered from the axis of z, while in the second case we shall assume that r is small compared with k.

Case I. When k is small compared with r, let us expand $\tanh\alpha k$ in ascending powers of αk and obtain

$$I=\tfrac{1}{2}\int_0^\infty\left[\alpha^3k-\frac{\alpha^5k^3}{3}+\frac{2\alpha^7k^3}{15}\ldots\right][e^{-\alpha(k-c)}+e^{-\alpha(k+c)}]J_1(\alpha r)d\alpha.\quad\ldots\quad(4.1)$$

If we put

$$\left.\begin{array}{l} k-c=\rho_1\cos\theta_1,\ r=\rho_1\sin\theta_1,\ k+c=\rho_2\cos\theta_2,\ r=\rho_2\sin\theta_2, \\ \text{so that}\quad \rho_1=\sqrt{(k-c)^2+r^2}\ \text{and}\ \rho_2=\sqrt{(k+c)^2+r^2}, \end{array}\right\}\ \ldots \quad (4.2)$$

we know that (Gray and Mathews, 1922, 100)

$$\left.\begin{array}{l} \displaystyle\int_0^\infty a^n e^{-a(k-c)} J_m(ar)\,da=\frac{1}{\rho_1^{\,n+1}}\,P_n^m(\mu_1), \\ \displaystyle\int_0^\infty a^n e^{-a(k-c)} J_m(ar)\,da=\frac{1}{\rho_2^{\,n+1}}\,P_n^m(\mu_2), \end{array}\right\}\ \ldots \quad (4.3)$$

where $\mu_1=\cos\theta_1$, $\mu_2=\cos\theta_2$ and $P_n^m(\mu_1)$, $P_n^m(\mu_2)$ are the usual associated Legendre functions.

Utilizing the above results, we get

$$I=\frac{k}{2}\left\{\left[\frac{P_3{}^1(\mu_1)}{\rho_1{}^4}+\frac{P_3{}^1(\mu_2)}{\rho_2{}^4}\right]-\frac{k^2}{3}\left[\frac{P_5{}^1(\mu_1)}{\rho_1{}^6}+\frac{P_5{}^1(\mu_2)}{\rho_2{}^6}\right]+\ldots\right\}\ \ldots \quad (4.4)$$

It is evident that for large values of r and hence of ρ_1 and ρ_2 compared with k, we get sufficiently accurate values of I by retaining only a few terms of the series (4.4).

Case II. For large values of k, let us put

$$e^{-ak}\tanh ak=e^{-ak}-2e^{-3ak}+2e^{-5ak}-2e^{-7ak}\ldots\ldots\ldots$$

whence we get

$$\begin{aligned} I=\tfrac{1}{2}\int_0^\infty & a^2e^{-a(k-c)}-2a^2e^{-a(3k-c)}+2a^2e^{-a(5k-c)}+\ldots\ldots \\ & +a^2e^{-a(k+c)}-2a^2e^{-a(3k+c)}+2a^2e^{-a(5k+c)}\ \ldots\ \ldots\ \ldots\ ]J_1(ar)\,da.\ \ldots \quad (4.5) \end{aligned}$$

For evaluating this integral we use the result

$$\int_0^\infty a^2e^{-ab}J_1(ar)\,da=\frac{3br}{(b^2+r^2)^{\frac{5}{2}}}=\frac{3r}{b^4(1+r^2/b^2)^{\frac{5}{2}}}$$

Hence

$$\begin{aligned} I=\frac{3r}{2}\Bigg[& \frac{1}{(k-c)^4[1+r^2/(k-c)^2]^{\frac{5}{2}}}-\frac{2}{(3k-c)^4[1+r^2/(3k-c)^2]^{\frac{5}{2}}}\ldots\ldots \\ & +\frac{1}{(k+c)^4[1+r^2/(k+c)^2]^{\frac{5}{2}}}-\frac{2}{(3k+c)^4[1+r^2/(3k+c)^4]^{\frac{5}{2}}}\ldots\ldots\Bigg]\ \ldots \quad (4.6) \end{aligned}$$

For small values of r compared with k and c, a few terms of the above series will suffice for the determination of I with sufficient accuracy.

BENGAL ENGINEERING COLLEGE,
SIBPORE.

References

Gray and Mathews, 1922, *Treatise on Bessel Functions*, p. 100.

Passer, W., 1935, *Sitzungsberichte Akad. Wiss. Wien*, **144**, 307.

Sen, B., 1937, *Bull. Cal. Math. Soc.*, **29**, 41.

Sen, B., 1940, *Bull. Cal. Math. Soc*, **32**, 73.

A NOTE ON SONINE'S POLYNOMIALS I

By

K. Basu

(Communicated by the Secretary—Received, March 8, 1943)

Abstract: An attempt has been made to discuss some properties of Sonine's polynomial function: its construction from a generating function analogous to that for Laguerre polynomials; evaluation of certain definite integrals involving products of Sonine functions; establishment of a few recurring formulae; and lastly, a treatment is given for representation of the function in terms of contour integrals by attacking direct the equation itself which the function satisfies, without losing sight of the asymptotic form of this function.

Up till now proper attention has hardly been paid to a form of confluent hypergeometric functional equation which is satisfied by Sonine's polynomial function (Sonine, 1880). Certain properties of this function have been found to be useful in tackling wave-mechanical problems, especially, in the work of solutions of some complex equations that can be framed in Dirac and Schrödinger mechanics. In this connection the role played by a kindred function, *viz.*, the Laguerre polynomial is too well known. Although confluent hypergeometric function, in some form or other, has been studied exhaustively by several authors for over a long period, it is apropos to discuss Sonine's function in order that its properties may form a tool at the hands of students of Applied Mathematics and Physics. My next endeavour will be to supplement a few of its applications, in order to justify that this function ought to play a good part if it is found to be more advantageous.

I

The generating function of Sonine's polynomial written in a slightly modified form as

$$S_\lambda^n(z) = 1 - \frac{\binom{n}{1}}{\lambda_1} z + \frac{\binom{n}{2}}{\lambda_2} z^2 - \frac{\binom{n}{3}}{\lambda_3} z^3 \ldots (-)^n \frac{\binom{n}{n}}{\lambda_n} z^n, \qquad \ldots \quad (1.1)$$

where $\binom{n}{r} = \dfrac{n(n-1)\ldots(n-r+1)}{r!}$, $\lambda_r = \lambda(\lambda+1)\ldots(\lambda+r-1)$, and λ stands for any rational positive quantity, can be constructed thus

$$\psi(z,\, t) \equiv \frac{exp(-zt/1-t)}{(1-t)^\lambda} = \sum_{n=0}^{\infty} t^n \frac{\Gamma(\lambda+n)}{n!\Gamma(\lambda)} S_\lambda^n(z)\,; \quad (\,|\,t\,| < 1) \qquad \ldots \quad (1.2)$$

as the explicit form given by (1.1) of the function S can be verified easily. Further, it is clear from actual substitution that $S_\lambda^n(z) \equiv S$ satisfies the equation:

$$z\frac{d^2S}{dz^2} + (\lambda - z)\frac{dS}{dz} + nS = 0. \qquad \ldots \quad (1.3)$$

P 17586

Writing it in self-adjoint form we get

$$\frac{d}{dz}\left(e^{-z}z^{\lambda}\frac{dS}{dz}\right)+e^{-z}z^{\lambda-1}nS=0. \qquad \ldots \quad (1.4)$$

Another self-adjoint form for $S' \equiv S^{n'}_{\lambda}(z)$ will be

$$\frac{d}{dz}\left(e^{-z}z^{\lambda}\frac{dS'}{dz}\right)+e^{-z}z^{\lambda-1}n'S'=0. \qquad \ldots \quad (1.5)$$

Multiplying (1.4) and (1.5) by S' and S in order, and subtracting, we get

$$\frac{d}{dz}\left[e^{-z}z^{\lambda}\left(S'\frac{dS}{dz}-S\frac{dS'}{dz}\right)\right]+(n-n')e^{-z}z^{\lambda-1}SS'=0.$$

It follows as a consequence

$$\int_0^{\infty} e^{-z}z^{\lambda-1}S^{n}_{\lambda}(z)S^{n'}_{\lambda}(z)dz=\frac{1}{n'-n}\left|e^{-z}z^{\lambda}\left(S'\frac{dS}{dz}-S\frac{dS'}{dz}\right)\right|_0^{\infty}=0 \; (n \neq n'). \quad (1.6)$$

It may be noted that putting $\psi=e^{-\frac{1}{2}z}z^{\frac{\lambda}{2}}S^{n}_{\lambda}(z)$, equation (1.3) may be thrown into the form,

$$\frac{d^2\psi}{dz^2}+\left(\frac{n}{z}-I\right)\psi=0 \qquad \ldots \quad (1{\cdot}7)$$

where $I \equiv \frac{1}{4}-\frac{\lambda}{2z}+\frac{\lambda(\lambda-2)}{4z^2}$. Clearly ψ is Wittaker's function $W_{n+\frac{\lambda}{2},\, \frac{\lambda-1}{2}}(z)$.

II.

Since $$\frac{exp(-zt/1-t)}{(1-t)^{\lambda}}=\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}S^{n}_{\lambda}(z) \qquad (|t|<1)$$

and $$\frac{exp(-zs/1-s)}{(1-s)^{\mu}}=\sum_{n=0}^{\infty}\frac{t^m}{m!}\frac{\Gamma(\mu+m)}{\Gamma(\mu)}S^{m}_{\mu}(z) \qquad (|s|<1)$$

we get by multiplication, and lastly by multiplying by a factor $e^{-z}z^p$ and integrating between 0 and ∞ :

$$\int_0^{\infty}\frac{exp\left\{-z\,\frac{1-ts}{(1-t)(1-s)}\right\}z^p dz}{(1-t)^{\lambda}(1-s)^{\mu}}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n}{n!}\frac{s^m}{m!}\frac{\Gamma(\lambda+n)\Gamma(\mu+m)}{\Gamma(\lambda)\Gamma(\mu)}$$

$$\times\int_0^{\infty}e^{-z}z^pS^{n}_{\lambda}(z)\,S^{m}_{\mu}(z)dz. \qquad \ldots \quad (2.1)$$

Putting $k=(1-ts)/(1-t)(1-s)$, and observing that, in general,

$$\int_0^{\infty}e^{-ax}x^{\beta}dx=\Gamma(\beta+1)/a^{\beta+1} \qquad \ldots \quad (2.2)$$

we get, equating coefficients of $t^n s^m$ from both sides,

$$\int_0^\infty z^p e^{-z} S_\lambda^n(z) S_\mu^m(z)\,dz = \frac{n!\,m!\,\Gamma(\lambda)\Gamma(\mu)\Gamma(p+1)}{\Gamma(\lambda+n)\Gamma(\mu+m)} \times \{\text{coefficient of } t^n s^m \text{ in the expansion of } (1-t)^{p+1-\lambda}(1-s)^{p+1-\mu}(1-ts)^{-p-1}\}. \quad \ldots \quad (2.3)$$

(a) When $p+1 \geq \lambda$ and $\geq \mu$. This coefficient

$$= \sum_r (-)^r \binom{p+1-\lambda}{r} t^r . \sum_{r'} (-) \binom{p+1-\mu}{r'} . \sum_\tau (-)^\tau \binom{-p-1}{\tau} (ts)^\tau,$$

where $\binom{-p-1}{\tau} = (-)^\tau \dfrac{\Gamma(p+\tau+1)}{\Gamma(p+1)}$, $\binom{p+1-\lambda}{r} = \dfrac{\Gamma(p+2-\lambda)}{r!\,\Gamma(p-\lambda-r+2)}$,

$$\binom{p+1-\mu}{r'} = \frac{\Gamma(p+2-\mu)}{r'!\,\Gamma(p-\mu-r'+2)}.$$

And, since $r+\tau=n$, $r'+\tau=m$, the result (2.3) takes the form:

$$\int_0^\infty z^p e^{-z} S_\lambda^n(z) S_\mu^m(z)\,dz \left\{ \begin{matrix} p+1 \geq \lambda \\ > \mu \end{matrix} \right\} = (-)^{n+m} \frac{n!\,m!\,\Gamma(\lambda)\Gamma(\mu)\Gamma(p+2-\lambda)\Gamma(p+2-\mu)}{\Gamma(\lambda+n)\Gamma(\mu+m)}$$

$$\times \sum_{\tau=0}^{m \text{ or } n} \frac{\Gamma(p+\tau+1)}{(n-\tau)!\,(m-\tau)!\,\tau!\,\Gamma(p+z-\lambda-n+\tau)\Gamma(p+2-\mu-m+\tau)}, \quad \ldots \quad (2.4)$$

the summation extends to m or n, whichever is less.

(b) When $\lambda > p+1 > \mu$. This coefficient of $t^n s^m$

$$= \sum_r (-)^r \binom{-\lambda+p+1}{r} t^r . \sum_{r'} (-)^{r'} \binom{p+1-\mu}{r'} . \sum_\tau (-)^\tau \binom{-p-1}{\tau} (ts)^\tau.$$

Noting that $\binom{-\lambda+p+1}{r} = (-)^r \dfrac{\Gamma(\lambda-p+r-1)}{\Gamma(\lambda-p-1)r!}$, we write

$$\int_0^\infty z^p e^{-z} S_\lambda^n(z) S_\mu^m(z)\,dz \{\lambda > p+1 > \mu\} = \frac{n!\,m!\,\Gamma(\lambda)\Gamma(\mu)\Gamma(p+2-\mu)}{\Gamma(\lambda+n)\Gamma(\mu+m)\Gamma(\lambda-p-1)}$$

$$\times \sum_{\tau=0}^{m(<n)} (-)^{m-\tau} \frac{\Gamma(p+\tau+1)\Gamma(\lambda-p-1+n-\tau)}{(n-\tau)!\,(m-\tau)!\,\tau!\,\Gamma(p+2-\mu-m+\tau)}. \quad \ldots \quad (2.5)$$

(c) When $\lambda > \mu > p+1$.

$$\int_0^\infty z^p e^{-z} S_\lambda^n(z) S_\mu^m(z)\,dz = \frac{n!\,m!\,\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+n)\Gamma(\mu+m)\Gamma(\lambda-p-1)\Gamma(\mu-p-1)}$$

$$\times \sum_{\tau=0}^{m<n} \frac{\Gamma(p+\tau+1)\Gamma(\lambda-p-1+n-\tau)\Gamma(\mu-p-1+m-\tau)}{(n-\tau)!\,(m-\tau)!\,\tau!}. \quad \ldots \quad (2.6)$$

(d) When $\lambda=\mu$, $p=\lambda-1$. It is clear from (2.3) that

$$\int_0^\infty z^{\lambda-1} e^{-z} S_\lambda^n(z) S_\lambda^m(z)\,dz = 0 \qquad (n \neq m) \quad \ldots \quad (2.7)$$

and
$$\int_0^\infty z^{\lambda-1} e^{-z} S_\lambda^n(z) S_\lambda^n(z)\,dz = \frac{n!\,[\Gamma(\lambda)]^2}{\Gamma(\lambda+n)}\,. \qquad \ldots \quad (2.8)$$

(e) When $\lambda=\mu$, $p=\lambda$, we get, from either (2.3) or (2.4), the following:—

$$\int_0^\infty z^{\lambda} e^{-z} S_\lambda^n(z) S_\lambda^n(z)\,dz = \frac{(\lambda+2n)n!\,[\Gamma(\lambda)]^2}{\Gamma(\lambda+n)}\,, \qquad \ldots \quad (2.9)$$

$$\int_0^\infty z^{\lambda} e^{-z} S_\lambda^n(z) S_\lambda^{n+1}(z)\,dz = -\frac{(n+1)!\,[\Gamma(\lambda)]^2}{\Gamma(\lambda+n)}\,. \qquad \ldots \quad (2.10)$$

Still more general types of integrals can be handled in this manner:

$$\frac{e^{-\lambda_1 z} exp(-2\lambda_1 zt/1-t)}{(1-t)^{\lambda}} \equiv \frac{exp\left(-\lambda_1 z\,\dfrac{1+t}{1-t}\right)}{(1-t)^{\lambda}} = \sum_{n=0}^{\infty} e^{-\lambda_1 z}\,\frac{t^n}{n!}\,\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\,S_\lambda^n(2\lambda_1 z),$$

$$\frac{e^{-\lambda_2 z} exp(-2\lambda_2 zs/1-s)}{(1-s)^{\mu}} \equiv \frac{exp\left(-\lambda_2 z\,\dfrac{1+s}{1-s}\right)}{(1-s)^{\mu}} = \sum_{m=0}^{\infty} e^{-\lambda_2 z}\,\frac{s^m}{m!}\,\frac{\Gamma(\mu+m)}{\Gamma(\mu)}\,S_\mu^m(2\lambda_2 z).$$

Multiplying together and introducing a factor z^p and integrating between 0 and ∞, we get

$$\int_0^\infty z^p e^{-(\lambda_1+\lambda_2)z} S_\lambda^n(2\lambda_1 z) S_\mu^m(2\lambda_2 z)\,dz = \frac{n!\,m!\,\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+n)\Gamma(\mu+m)}$$

$$\times \text{coeff. of } t^n s^m \text{ in } \int_0^\infty \frac{z^p e^{-\Lambda z}\,dz}{(1-t)^{\lambda}\,(1-s)^{\mu}}\,, \text{ where } \Lambda \equiv \lambda_1\left(\frac{1+t}{1-t}\right)+\lambda_2\left(\frac{1+s}{1-s}\right)$$

$$= \text{coeff. of } t^n s^m \text{ in } \frac{n!\,m!\,\Gamma(\lambda)\Gamma(\mu)\Gamma(p+1)}{\Gamma(\lambda+n)\Gamma(\mu+m)}\;\frac{1}{(1-t)^{\lambda}\,(1-s)^{\mu}\,\Lambda^{p+1}}\,. \qquad \ldots \quad (2.11)$$

Now $$\frac{1}{(1-t)^{\lambda}\,(1-s)^{\mu}}\cdot\frac{1}{\Lambda^{p+1}} = \frac{1}{(\lambda_1+\lambda_2)^{p+1}}\cdot\frac{(1-t)^{p+1-\lambda}\,(1-s)^{p+1-\mu}}{[1+\theta(t-s)-ts]^{p+1}}\,,$$

where $\theta=(\lambda_1-\lambda_2)/(\lambda_1+\lambda_2)$.

Using multinomial theorem we find the general term of $[\;\;]^{p+1}$ in the above, which means

$$[1+\theta(t-s)-ts]^{-p-1} = \sum_{r=0}^{\infty}(-)^r \sum_\beta \sum_\gamma \sum_\delta \frac{\Gamma(p+r+1)}{\Gamma(p+1)\beta!\,\gamma!\,\delta!}\,(\theta t)^{\beta}(-\theta s)^{\gamma}(-ts)^{\delta}$$

$$= \sum_{r=0}^{\infty}\sum_\beta \sum_\gamma \sum_\delta (-)^{r+\gamma+\delta}\,\frac{\Gamma(p+r+1)}{\Gamma(p+1)\beta!\,\gamma!\,\delta!}\,\theta^{\beta+\gamma} t^{\beta+\delta} s^{\gamma+\delta} \qquad (\beta+\gamma+\delta=r) \quad (2.12)$$

Considering the case $p+1 \geq \lambda > \mu$, the co-efficient stands thus

$$\sum_{r_1}\sum_{r_2}\sum_{r}\sum_\beta\sum_\gamma\sum_\delta (-)^{r_1+r_2+r+\gamma+\delta}\,\frac{n!\,m!\,\Gamma(\lambda)\Gamma(\mu)\Gamma(p+1)}{(\lambda_1+\lambda_2)^{p+1}\Gamma(\lambda+n)\Gamma(\mu+m)}\,\theta^{\beta+\gamma}$$

$$\times \binom{p+1-\lambda}{r_1}\binom{p+1-\mu}{r_2}\frac{\Gamma(p+r+1)}{\Gamma(p+1)\beta!\,\gamma!\,\delta!}\,t^{r_1+\beta+\delta} s^{r_2+\gamma+\delta}\,, \qquad \ldots \quad (2.13)$$

Out of six unknowns, *viz.*, $\beta, \gamma, \delta, r_1, r_2, r$, we have only three relations *viz.*, $r_1+\beta+\delta=n$, $r_2+\gamma+\delta=m$ and $\beta+\gamma+\delta=r$. Hence we see that

$$\int_0^\infty z^p e^{-(\lambda_1+\lambda_2)z} \mathrm{S}_\lambda^n(2\lambda_1 z)\mathrm{S}_\mu^m(2\lambda_2 z)dz \left\{\begin{matrix} p+1>\lambda; & n\neq m \\ >\mu; & \lambda_1\neq\lambda_2 \end{matrix}\right\}$$

$$=\sum_{\beta=0}^{n}\sum_{\gamma=0}^{m}\sum_{\delta=0}^{m(<n)}(-)^{m+n+\gamma}\theta^{\beta+\gamma}\frac{n!m!\Gamma(\lambda)\Gamma(\mu)\Gamma(p+1+\beta+\gamma+\delta)}{(\lambda_1+\lambda_2)^{p+1}\beta!\gamma!\delta!\Gamma(\lambda+n)\Gamma(\mu+m)}$$

$$\times\binom{p+1-\lambda}{n-\beta-\delta}\binom{p+1-\mu}{m-\gamma-\delta}. \qquad \ldots \quad (2.14)$$

III

Since $$\frac{exp(-zt/1-t)}{(1-t)^\lambda}=\frac{exp(-zt/1-t)}{(1-t)^\mu}\times\frac{1}{(1-t)^{\lambda-\mu}} \qquad \left(\begin{matrix}\lambda>\mu\\ |t|<1\end{matrix}\right)$$

we have $$\sum_{n=0}^\infty \frac{t^n}{n!}\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\mathrm{S}_\lambda^n(z)=\sum_{m=0}^\infty\frac{t^m}{m!}\frac{\Gamma(\mu+m)}{\Gamma(\mu)}\mathrm{S}_\mu^m(z).\sum_{r=0}^\infty\frac{t^r}{r!}\frac{\Gamma(\lambda-\mu+r)}{\Gamma(\lambda-\mu)}$$

$$=\sum_{m=0}^\infty\sum_{r=0}^\infty t^{m+r}\frac{\Gamma(\mu+m)\Gamma(\lambda-\mu+r)}{m!r!\Gamma(\mu)\Gamma(\lambda-\mu)}\mathrm{S}_\mu^m(z).$$

If $m+r=n$, equating co-efficients of t^n we derive:

$$\mathrm{S}_\lambda^n(z)=\frac{n!\Gamma(\lambda)}{\Gamma(\lambda+n)\Gamma(\mu)\Gamma(\lambda-\mu)}\sum_{r=0}^\infty\frac{\Gamma(\mu+n-r)\Gamma(\lambda-\mu+r)}{(n-r)!r!}\mathrm{S}_\mu^{n-r}(z). \qquad \ldots \quad (3.1)$$

From Eq. (2.4) we get when $\lambda=\mu$ $(m\leq n)$

$$\int_0^\infty z^p e^{-z}\mathrm{S}_\lambda^n(z)\mathrm{S}_\lambda^m(z)dz=(-)^{m+n}\frac{[\Gamma(\lambda)\Gamma(p+2-\lambda)]^2}{\Gamma(\lambda+n)\Gamma(\lambda+m)}$$

$$\times\sum_{r=0}^m\frac{\Gamma(p+1+r)}{(n-r)!(m-r)!r!\Gamma(p+2-\lambda-n+r)\Gamma(p+2-\lambda-m+r)}. \qquad \ldots \quad (3.2)$$

We observe that $p=\lambda-2, \lambda-3, \ldots$ give zero values in each case.

(A) When $p=\lambda-1$:

$$\int_0^\infty z^{\lambda-1}e^{-z}\mathrm{S}_\lambda^n(z)\mathrm{S}_\lambda^m(z)dz=(-)^{n+m}\frac{n![\Gamma(\lambda)]^2}{\Gamma(\lambda+n)(n-m)!(m-n)!}. \qquad \ldots \quad (3.3)$$

The only possible contribution comes from $n=m$, and we get result (2.8), although result (2.7) is self-evident.

(B) When $p=\lambda$:

$$\int_0^\infty z^\lambda e^{-z}\mathrm{S}_\lambda^n(z)\mathrm{S}_\lambda^m(z)dz=(-)^{m+n}\frac{n![\Gamma(\lambda)]^2\{\lambda(n-m+1)+2m\}}{\Gamma(\lambda+n)(n-m+1)!(m+1-n)!}, \qquad \ldots \quad (3.4)$$

summing up the series in the reverse order, *i.e.*, starting from $r=m, m-1, \ldots$ Under this we get the following cases: $n=m$ and $n=m+1$, already dealt with, and

$$\int_0^\infty z^\lambda e^{-z}\mathrm{S}_\lambda^n(z)\mathrm{S}_\lambda^m(z)dz \quad \{n-m\geq 2\}=0. \qquad \ldots \quad (3.5)$$

(C) . When $p=\lambda+1$:

$$\int_0^\infty z^{\lambda+1}e^{-z}S_\lambda^n(z)S_\lambda^m(z)dz = (-)^{n+m}\frac{2[\Gamma(\lambda)]^2 n!\{K\}}{\Gamma(\lambda+n)(n-m+2)!(m-n+2)!}, \quad \ldots \quad (3.6)$$

where $K \equiv \lambda^2[(n-m)^2+3(n-m)+2]+\lambda[(n-m)^2+3(2m+1)(n-m)$
$+2(6m+1)]+6m(n+m)$.

We get altogether *three* cases having non-zero values, *viz.*, when $n=m$, $m+1$ and $m+2$. Thus

$$\int_0^\infty z^{\lambda+1}e^{-z}S_\lambda^n(z)S_\lambda^n(z)dz = \frac{n![\Gamma(\lambda)]^2}{\Gamma(\lambda+n)}\{\lambda^2+(6n+1)\lambda+6n^2\} \quad \ldots \quad (3.7)$$

$$\int_0^\infty z^{\lambda+1}e^{-z}S_\lambda^n(z)S_\lambda^{n-1}(z)dz = -\frac{n![\Gamma(\lambda)]^2 2(\lambda+2n-1)}{\Gamma(\lambda+n-1)} \quad \ldots \quad (3.8)$$

$$\int_0^\infty z^{\lambda+1}e^{-z}S_\lambda^n(z)\,S_\lambda^{n-2}(z)dz = \frac{n![\Gamma(\lambda)]^2}{\Gamma(\lambda+n-2)} \quad \ldots \quad (3.9)$$

and, of course, $\displaystyle\int_0^\infty z^{\lambda+1}e^{-z}S_\lambda^n(z)S_\lambda^m(z)dz\{n-m \geq 3\} = 0. \quad \ldots \quad (3.10)$

Let us evaluate another type of integrals (when the lower suffix λ varies) with the help of above results and result (3.1).

(D) When $\lambda-\mu=1$. $(n=m)$

(*a*) $p=\lambda-2$.

$$\int_0^\infty z^{\lambda-2}e^{-z}\;S_\lambda^n(z)\,S_{\lambda-1}^n(z)dz$$

$$=\int_0^\infty z^{\lambda-2}e^{-z}\;S_{\lambda-1}^n(z)\left\{\frac{n!\Gamma(\lambda)}{\Gamma(\lambda+n)\Gamma(\lambda-1)}\times\sum_{\tau=0}^{n}\frac{\Gamma(\lambda-1+n-\tau)}{(n-\tau)!}\;S_{\lambda-1}^{n-\tau}(z)\right\}dz,$$

taking over the series for $S_\lambda^n(z)$ from ... (3.1)

The only contribution to the integral, from the series, is (when $\tau=0$) the factor of $S_{\lambda-1}^n(z)$. Hence, after reduction,

$$\int_0^\infty z^{\lambda-2}e^{-z}S_\lambda^n(z)S_{\lambda-1}^n(z)dz = \frac{n!\Gamma(\lambda)\Gamma(\lambda-1)}{\Gamma(n+\lambda)}. \quad \ldots \quad (3.11)$$

(*b*) $p=\lambda-1$.

$$\int_0^\infty z^{\lambda-1}e^{-z}S_\lambda^n(z)S_{\lambda-1}^n(z)dz=\int_0^\infty z^{\lambda-1}e^{-z}S_{\lambda-1}^n(z)\left\{\sum_\tau(\;)\,S_{\lambda-1}^{n-\tau}(z)\right\}dz,$$

where the series, for $S_\lambda^n(z)$ is included within { }. The contributions come from $S_{\lambda-1}^n$, $S_{\lambda-1}^{n-1}$; others vanish. Hence

$$\int_0^\infty z^{\lambda-1}e^{-z}S_\lambda^n(z)S_{\lambda-1}^n(z)dz = \frac{n!\Gamma(\lambda)\Gamma(\lambda-1)}{\Gamma(n+\lambda-1)}. \quad \ldots \quad (3.12)$$

(c) $p=\lambda$.

$$\int_0^\infty s^\lambda e^{-s}\, S^n_\lambda(s) S^n_{\lambda-1}(s)ds = \frac{n\,!\Gamma(\lambda)\Gamma(\lambda-1)}{\Gamma(\lambda+n-1)}(\lambda+3n). \qquad \dots \quad (3.13)$$

(E) When $\lambda-\mu=2$.

We quote the result for $p=\lambda-1$ only.

$$\int_0^\infty s^{\lambda-1} e^{-s}\, S^n_{\lambda-2}(s)\, S^n_\lambda(s)ds = \frac{n\,!\Gamma(\lambda)\Gamma(\lambda-2)}{\Gamma(\lambda+n-2)}, \qquad \dots \quad (3.14)$$

and consequently

$$\int_0^\infty s^\lambda e^{-s}\, S^n_{\lambda-1}(s)\, S^n_{\lambda+1}(s)ds = \frac{n\,!\Gamma(\lambda+1)\Gamma(\lambda-1)}{\Gamma(\lambda+n-1)}. \qquad \dots \quad (3.15)$$

IV

Since
$$\frac{exp(-st/1-t)}{(1-t)^{\lambda+1}} = \sum_{n=0}^\infty \frac{t^n}{n!}\frac{\Gamma(\lambda+n+1)}{\Gamma(\lambda+1)} S^n_{\lambda+1}(s)$$

and
$$\frac{exp(-st/1-t)}{(1-t)^\lambda} = \sum_{n=0}^\infty \frac{t^n}{n!}\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} S^n_\lambda(s),$$

it is clear that

$$(1-t)\sum_n \frac{t^n}{n!}\frac{\Gamma(\lambda+n+1)}{\Gamma(\lambda+1)} S^n_{\lambda+1}(s) = \sum_n \frac{t^n}{n!}\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} S^n_\lambda(s). \qquad \dots \quad (4.1)$$

Equating coefficients of t^n from both sides we derive:

$$(n+\lambda)S^n_{\lambda+1}(s) - n\, S^{n-1}_{\lambda+1}(s) = \lambda\, S^n_\lambda(s). \qquad \dots \quad (4.2)$$

Again
$$\frac{d}{dt}\frac{exp(-st/1-t)}{(1-t)^\lambda} = \frac{\lambda exp(-st/1-t)}{(1-t)^{\lambda+1}} - \frac{s exp(-st/1-t)}{(1-t)^{\lambda+2}}. \qquad \dots \quad (4.3)$$

and
$$\frac{d}{dt}\left[(1-t)\frac{exp(-st/1-t)}{(1-t)^{\lambda+1}}\right]$$
$$= -\frac{exp(-st/1-t)}{(1-t)^{\lambda+1}} + (1-t)\frac{d}{dt}\frac{exp(-st/1-t)}{(1-t)^{\lambda+1}}. \qquad \dots \quad (4.4)$$

Equating the right-hand members of (4.3) and (4.4) we get

$$\frac{s\, exp(-st/1-t)}{(1-t)^{\lambda+2}} = (\lambda+1)\frac{exp(-st/1-t)}{(1-t)^{\lambda+1}} - (1-t)\frac{d}{dt}\frac{exp(-st/1-t)}{(1-t)^{\lambda+1}};$$

which gives

$$s\sum \frac{t^n}{n!}\frac{\Gamma(n+\lambda+2)}{\Gamma(\lambda+2)} S^n_{\lambda+2}(s) = (\lambda+1)\sum \frac{t^n}{n!}\frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda+1)} S^n_{\lambda+1}(s)$$
$$-(1-t)\sum \frac{t^{n-1}}{(n-1)!}\frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda+1)} S^n_{\lambda+1}(s). \qquad \dots \quad (4.5)$$

Equating co-efficients of t^n we derive ultimately:

$$zS^{n}_{\lambda+2}(z) = (\lambda+1)\left[S^{n}_{\lambda+1}(z) - S^{n+1}_{\lambda+1}(z)\right], \qquad \ldots \quad (4.6)$$

that is to say,

$$\left.\begin{aligned} zS^{n}_{\lambda}(z) &= (\lambda-1)[S^{n}_{\lambda-1}(z) - S^{n+1}_{\lambda+1}(z)] \\ \text{and} \quad zS^{n}_{\lambda+1}(z) &= \lambda[S^{n}_{\lambda}(z) - S^{n+1}_{\lambda}(z)] \end{aligned}\right\} \qquad \ldots \quad (4.7)$$

Eqs. (4.2) and (4.7) give

$$zS^{n}_{\lambda}(z) = (2n+\lambda)S^{n}_{\lambda}(z) - (n+\lambda)S^{n+1}_{\lambda}(z) - nS^{n-1}_{\lambda}(z). \qquad \ldots \quad (4.8)$$

Again, $$\frac{d}{dz}\,\frac{exp(-zt/1-t)}{(1-t)^{\lambda}} = \sum_{n}\frac{t^n}{n!}\,\frac{\Gamma(n+\lambda)}{\Gamma(\lambda)}\,\frac{d}{dz}S^{n}_{\lambda}(z).$$

But the left-hand member

$$= \frac{(1-t)-1}{1-t}\cdot\frac{exp(-zt/1-t)}{(1-t)^{\lambda}} = \frac{exp(-zt/1-t)}{(1-t)^{\lambda}} - \frac{exp(-zt/1-t)}{(1-t)^{\lambda+1}}.$$

$$\therefore \quad \sum_{n}\frac{t^n}{n!}\,\frac{\Gamma(n+\lambda)}{\Gamma(\lambda)}S^{n}_{\lambda}(z) - \sum_{n}\frac{t^n}{n!}\,\frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda+1)}\,S^{n}_{\lambda+1}(z) = \sum\frac{t^n}{n!}\,\frac{\Gamma(n+\lambda)}{\Gamma'\lambda)}\,\frac{d}{dz}\,S^{n}_{\lambda}(z).$$

Equating coefficients of t^n and simplifying, we obtain

$$\frac{d}{dz}\,S^{n}_{\lambda}(z) = S^{n}_{\lambda}(z) - \frac{n+\lambda}{\lambda}\,S^{n}_{\lambda+1}(z), \qquad \ldots \quad (4.9)$$

whence, we get, after utilising (4.7) and (4.8):

$$z\,\frac{d}{dz}\,S^{n}_{\lambda}(z) = n\,\{S^{n}_{\lambda}(z) - S^{n-1}_{\lambda}(z)\} \qquad \ldots \quad (4.10)$$

Again from (4.2), the above takes the form:

$$z\,\frac{d}{dz}\,S^{n}_{\lambda}(z) = (\lambda-1)\{S^{n}_{\lambda-1}(z) - S^{n}_{\lambda}(z)\}. \qquad \ldots \quad (4.11)$$

From (4.8) and (4.10) we derive:

$$z\,\frac{d}{dz}\,S^{n}_{\lambda}(z) = (z-n-\lambda)\,S^{n}_{\lambda}(z) + (n+\lambda)S^{n+1}_{\lambda}(z). \qquad \ldots \quad (4.12)$$

V

We have seen that the differential equation (1.3) is satisfied by a polynomial function $S^{n}_{\lambda}(z)$ given by (1.1). We may attempt at seeking solutions of (1.3) by tackling the problem apriori. If we write a series for the solution of (1.3), as

$$S = \sum_{\nu=0}^{\infty} c_{\nu}\,z^{\rho+\nu}, \qquad \ldots \quad (5.1)$$

the indicial equation becomes $c_0\rho(\rho+\lambda-1)=0$, and the relation between the coefficients is given by

$$c_{\nu+1}=c_\nu\cdot\frac{\rho+\nu-n}{(\rho+\nu+1)(\rho+\nu+\lambda)}\cdot \qquad (\nu=0,\ 1,\ 2,\ldots) \qquad \ldots \quad (5.2)$$

When $\rho=0$, $c_{\nu+1}=c_\nu\,\dfrac{\nu-n}{(\nu+1)(\nu+\lambda)}$; the series terminates when $\nu=n$.

When $\rho=1-\lambda$, $c_{\nu+1}=c_\nu\cdot\dfrac{1+\nu-\lambda-n}{(2+\nu-\lambda)(\nu+1)}$; the series cannot terminate as λ is not an integer, but extends to ∞. Hence we get the following series as solutions in the vicinity of $z=0$:—

$$\left.\begin{aligned}&(i)\quad c_0\left[1-\frac{n}{1.\lambda}z+\frac{n(n-1)}{2!\lambda(\lambda+1)}z^2-\ldots(\ )z^n\right]\\&(ii)\quad z^{1-\lambda}c_0'\left[1+\frac{n+\lambda-1}{1.(\lambda-2)}z+\frac{(n+\lambda-1)(n+\lambda-2)}{2!(\lambda-2)(\lambda-3)}z^2+\ \ldots.\ \infty\right].\end{aligned}\right\}\quad (5.3)$$

Putting $c_0=1$ in (*i*) we derive Sonine's polynomial. The solution (*ii*) is valid for negative values of λ and finite values of z, in order that it may be regular near $z=0$. Since λ is taken positive, we discard this solution.

Next, let us seek a solution in the form of a contour integral. For this, let us put

$$S=z^{1-\lambda}V(z). \qquad \ldots \quad (5.3')$$

We get the following equation for $V(z)$:

$$z\frac{d^2V}{dz^2}+(2-\lambda-z)\ \frac{dV}{dz}+(n+\lambda-1)V=0. \qquad \ldots \quad (5.4)$$

To apply Jordan's method of solution we put the above in the form:

$$Q(z)V''(z)-\lambda_1Q'(z)V'(z)-R(z)V'(z)+(\lambda_1+1)R'(z)V(z)=0, \quad \ldots \quad (5.5)$$

where the dashes denote differential coefficients with regard to z, and

$$Q(z)=z,\ Q'(z)=1,\ (\lambda_1+1)R'(z)=n+\lambda-1,\ \lambda_1+R(z)=\lambda+z-2.$$

Since $R'(z)=1$, we have $\lambda_1=n+\lambda-2$ and $R(z)=z-n$.

The function $V=\displaystyle\int_c\phi(\zeta)(\zeta-z)^{\lambda_1+1}d\zeta$ will be a solution of (5.4) (MacRobert), provided $\phi(\zeta)$ satisfies

$$\phi(\zeta)R(\zeta)=\frac{d}{d\zeta}\{\phi(\zeta)Q(\zeta)\}, \qquad \ldots \quad (5.6)$$

and consequently

$$\int_c\frac{d}{d\zeta}\left\{\phi(\zeta)Q(\zeta)(\zeta-z)^{\lambda_1}\right\}d\zeta=0. \qquad (5.7)$$

CALCUTTA UNIVERSITY CENTRAL LIBRARY

remains to be satisfied by choosing *suitable* contour, so that the function within { } in (5.7) gives equal values at the initial and final points of the contour. Eq. (5.6), when worked out, gives on integration

$$\phi(\zeta) \sim \zeta^{-n-1} e^{\zeta}$$

to be multiplied by a suitable constant afterwards. Hence

$$V \sim \int_c e^{\zeta} \zeta^{-n-1} (\zeta - z)^{n+\lambda-1} d\zeta. \qquad \ldots \quad (5.8)$$

If we choose $\zeta = -t$, the above is thrown into the form:

$$V = \text{Const.} \times \int_c e^{-t} (-t)^{-n-1} (z+t)^{n+\lambda-1} dt. \qquad \ldots \quad (5.9)$$

Hence S can be written as

$$S = \text{Const.} \times z^{1-\lambda} \int_{\infty}^{(0+)} e^{-t} (-t)^{-n-1} (z+t)^{n+\lambda-1} dt, \qquad \ldots \quad (5.10)$$

where arg z has its principal value, and the path of integration starts at ∞ on the real axis, encircles the origin in the (+) *ve* direction and goes back to the initial point; so that the branch point $t = -z$ is outside the contour. Further, the integrand is made one-valued by taking $|\arg(-t)| < \pi$, and taking that value of $\arg(t+z)$ which $\rightarrow 0$ as $t \rightarrow \infty$, by a path lying inside the contour. Under these conditions the integrand is an analytic function of z.

Since n is an integer, we can write finally:

$$S(z) = \frac{n!\,\Gamma(\lambda)}{\Gamma(\lambda+n)} \frac{1}{2\pi i} z^{1-\lambda} \int_{\infty}^{(0+)} e^{-t} t^{-n-1} (z+t)^{n+\lambda-1} dt, \qquad \ldots \quad (5.11)$$

where the Const. in (5.10) $= (-)^{n+1} n!\,\Gamma(\lambda)/\Gamma(\lambda+n)$.

This integral tends to a finite value at $t \rightarrow \infty$. This behaviour can be judged thus. The integral

$$= z^{n+\lambda-1} \int_c e^{-t} t^{-n-1} \left(1 + \frac{t}{z}\right)^{n+\lambda-1} dt$$

$$= z^{n+\lambda-1} \left\{ \int_c \frac{e^{-t}\, dt}{t^{n+1}} + \binom{n+\lambda-1}{1} \frac{1}{z} \int_c \frac{e^{-t} dt}{t^{n}} + \ldots \right\}$$

where the integration is to be carried out over the loop defined already. This would have given a divergent (or semi-convergent) series if the series within { } extended to ∞ (for here $|t| > z$), but, on account of the integral nature of n, the series cannot extend to ∞, for Cauchy's theorem gives the values of these n integrals, other integrals give zero values in each case. Since $t = 0$ is the only singularity (pole) inside the contour, we can as well define the contour by choosing any one enclosing it, not necessarily extending to ∞. Eq. (5.7) is now clearly satisfied, *viz.*,

$$\int_{\infty}^{(0+)} \frac{d}{dt}\left\{e^{-t}t^{-n}(t+z)^{n+\lambda-2}\right\}dt = 0, \qquad \ldots \quad (5.12)$$

as, under the circumstances, the expression within { } is single valued at terminal points of the contour.

To evaluate (5.11) we deform the contour to a small circle round $t=0$ and collect the residues. Thus

$$S_\lambda^n(z) = \frac{n!\Gamma(\lambda)}{\Gamma(\lambda+n)}\frac{1}{2\pi i}\, z^{1-\lambda}\int_c \frac{e^{-t}}{t^{n+1}} z^{n+\lambda-1}\left(1+\frac{t}{z}\right)^{n+\lambda-1} dt$$

$$= \frac{n!\Gamma(\lambda)}{\Gamma(\lambda+n)}\cdot\frac{1}{2\pi i} z^n \left[\int_c \frac{e^{-t}dt}{t^{n+1}} + \binom{n+\lambda-1}{1}\frac{1}{z}\int_c \frac{e^{-t}}{t^n} + \ldots \right.$$

$$\left. + \binom{n+\lambda-1}{n}\frac{1}{z^n}\int \frac{e^{-t}}{t}\, dt + 0\left(\frac{1}{z^{n+1}}\right)\right].$$

Since the residue contributions from $0\left(\frac{1}{z^{n+1}}\right)$ are nil, we get the series up to nth term, and it is the Sonine's polynomial.

At this point we may construct some recurring formulae. Result (5.12) gives after differentiation of integrand

$$n\int_c e^{-t}t^{-n-1}(t+z)^{n+\lambda-2}dt + \int_c e^{-t}t^{-n}(t+z)^{n+\lambda-2}dt$$

$$-(n+\lambda-2)\int_c e^{-t}t^{-n}(t+z)^{n+\lambda-3}dt = 0. \ \ldots \quad (5.13)$$

Remembering

$$S_{\lambda-1}^n(z) = \frac{n!\Gamma(\lambda-1)}{\Gamma(n+\lambda-1)}\cdot\frac{1}{2\pi i}\, z^{2-\lambda}\int_c t^{-n-1}e^{-t}(t+z)^{n+\lambda-2}dt,$$

$$S_\lambda^{n-1}(z) = \frac{(n-1)!\Gamma(\lambda)}{\Gamma(n+\lambda-1)}\cdot\frac{1}{2\pi i}\, z^{1-\lambda}\int_c t^{-n}e^{-t}(t+z)^{n+\lambda-2}dt,$$

$$S_{\lambda-1}^{n-1}(z) = \frac{(n-1)!\Gamma(\lambda-1)}{\Gamma(n+\lambda-2)}\cdot\frac{1}{2\pi i}\, z^{2-\lambda}\int_c t^{-n}e^{-t}(t+z)^{n+\lambda-3}dt,$$

we get, by multiplying (5.15) by the factor

$$z^{2-\lambda}\frac{(n-1)!\Gamma(\lambda-1)}{\Gamma(n+\lambda-1)}\cdot\frac{1}{2\pi i},$$

the following relation:

$$zS_\lambda^{n-1}(z) = (\lambda-1)\{S_{\lambda-1}^{n-1}(z) - S_{\lambda-1}^{n}(z)\},$$

derived already in Sec. 4. Similarly, others can be derived. Thus

$$S_\lambda^n(z) = \frac{n!\Gamma(\lambda)}{\Gamma(\lambda+n)}\cdot\frac{1}{2\pi i} z^{-\lambda}\int_c e^{-t}t^{-n-1}(t+z)^{n+\lambda-1}[(t+z)-t]dt$$

$$= \frac{n+\lambda}{\lambda} S^{n}_{\lambda+1}(z) - \frac{n}{\lambda} S^{n-1}_{\lambda+1}(z)$$

or

$$\lambda S^{n}_{\lambda}(z) = (n+\lambda) S^{n}_{\lambda+1}(z) - n S^{n-1}_{\lambda+1}(z).$$

Again,

$$\frac{d}{dz} S^{n}_{\lambda}(z) = \frac{1}{2\pi i} \frac{n!\Gamma(\lambda)}{\Gamma(\lambda+n)} \left[\left(\frac{1-\lambda}{z}\right) z^{1-\lambda} \int_c t^{-n-1} e^{-t} (t+z)^{n+\lambda-1} dt \right]$$

$$+ \frac{1}{2\pi i} \frac{n!\Gamma(\lambda-1)}{\Gamma(\lambda+n-1)} \left[\left(\frac{\lambda-1}{z}\right) z^{2-\lambda} \int_c t^{-n-1} e^{-t} (t+z)^{n+\lambda-2} dt \right.$$

$$= \frac{1-\lambda}{z} S^{n}_{\lambda}(z) + \frac{\lambda-1}{z} S^{n}_{\lambda-1}(z).$$

$$\therefore \quad z \frac{d}{dz} S^{n}_{\lambda}(z) = (\lambda-1)[S^{n}_{\lambda-1}(z) - S^{n}_{\lambda}(z)].$$

Following Whittaker and Watson (1927, § 16.3) the asymptotic expansion for $S^{n}_{\lambda}(z)$ for large values of $|z|$ can be written down, which in a simplified form stands thus:

$$S^{n}_{\lambda}(z) \sim z^{n} \left\{ 1 + \sum_{r=1}^{\infty} (-)^{r} \binom{n}{r} \frac{\Gamma(\lambda+n)}{\Gamma(n+\lambda-r)} \frac{1}{z^{r}} \right\}, \qquad \ldots \quad (5.14)$$

where $|\arg z| \leqq \pi - \alpha < \pi$.

My best thanks are due to Prof. S. N. Bose, Dacca University, who inspired me a few years back in this line of work.

Department of Applied Mathematics,
University of Calcutta.

References

MacRobert, T. M., 1933, *Functions of a Complex Variable*, §95.

Sonine, N., 1880, *Math. Ann*, 16, 41.

Whittaker, E. T. and Watson, G. N., 1927, *A Course of Modern Analysis*, 4th ed.

REVIEW

Algebra, Volume I.—By F. W. Levi, Hardinge Professor of Mathematics, Calcutta University. Published by the University of Calcutta, 1942. Pp. xii + 304. Rs. 5.

The interest taken in Modern Algebra has steadily been growing in India, especially since its introduction in the M.Sc. course in Pure Mathematics of the Calcutta University, six years ago. The publication of Professor Levi's book is most opportune, and sure to stimulate further interest in this important branch of Mathematics. Already his lecture notes on the subject, of which the present book is a fully revised and enlarged edition, have (besides serving as a basis for class-teaching in the Post-Graduate department of the Calcutta University) awakened a number of Indian Statisticians to the possibilities of the application of Finite Algebra to the theory of "Design of Experiments." It is to be hoped that the present book would be found useful both by those who want to study Modern Algebra for its own sake, and by those who would like to apply it to the problems of Physics, Applied Mathematics or Statistics.

The volume under review covers six chapters. The first chapter deals with systems of linear equations. The notions of n-vector and vector space are introduced and applied to the solution of both homogeneous and non-homogeneous systems of equations. The notion of a matrix is introduced and the connection between the rank of a matrix and a vector space established. Properties of determinants and matrix products are discussed.

The second chapter covers in a nutshell the main theorems of General Algebra. Thus the essential properties of modules, rings (including theorems on factorisation) and fields (including theorems on extension) are developed. It is pointed out that the major part of the theorems of Chapter I (*viz.*, excluding those depending on orthogonality) remain valid when the co-ordinates of an n-vector belong to an arbitrary field.

In the third chapter the theorems of General Algebra are applied to the study of certain specific topics including cyclotomic polynomials, Galois fields, irreducibility of polynomials, symmetric polynomials, solution of cubic and bi-quadratic equations, and resultants. The fundamental theorem of classical Algebra asserting the closedness of the field of complex numbers is derived as a special case of a more general theorem.

The fourth chapter deals with the properties of continued fractions. The subject is treated from a point of view more general than that adopted in usual text-books.

The fifth chapter is concerned with the approximation of the roots of equations with real or complex co-efficients. Besides Horner's, Lagrange's, Newton's and Graffe's methods of approximating to the roots, the theorems of Fourier-Budan, Sturm, Kakeya, Poulain and Gauss on the location of the roots on the real axis or the complex plane are given.

In the sixth and last chapter, the essential theorems of the theory of matrices have been developed including those on the transformation of matrices to the canonical form, elementary divisors, and symmetric and Hermitian matrices. The properties of Hermitian matrices are applied to the theory of quadratic and bilinear forms.

Professor Levi has succeeded in giving, in a brief compass of about 300 pages, an admirable survey of a large portion of Modern Algebra. The appearance of the second volume of the present work, in which he promises to deal with Groups and Ideals, would be eagerly awaited, and there is no doubt that its publication would be widely welcomed.

R. C. BOSE

CORRECTIONS

VOL. 34, NO. 4

Page	*line*	*for*	*read*
187	12	2μ	2^{μ}
188	9	$d\chi$	dx
,,	19	$\Sigma n^2 a_n \alpha_{\mu+1}(nt) = \frac{\phi'_{\mu}(t)}{t}$	$\Sigma n^2 a_n \alpha_{\mu+1}(nt)$
,,	20	$t\phi'_{\mu}(t)$	$t^2 \Sigma n^2 a_n \alpha_{\mu+1}(nt)$
189	3, 4	*Omit completely*	
,,	6	$t\phi'_{\mu}(t)$	$t^2 \Sigma n^2 a_n \alpha_{\mu+1}(nt)$
,,	15	$\leqq n+1$	$< n+1$
192	8	$h+r$	$h-r$
194	24	$a > \mu+\frac{1}{2}$	$r > \mu+\frac{1}{2}$
196	6	$0\left(\frac{1}{2\mu+2}\right)$	$0\left(\frac{1}{t^{2\mu+2}}\right)$

CALCUTTA MATHEMATICAL SOCIETY

RECEIPTS AND DISBURSEMENTS ACCOUNTS FOR THE YEAR ENDING 31st DECEMBER, 1942

RECEIPTS

	Rs.	A.	P.	Rs.	A.	P.
1. To Opening Balance :—						
(a) Cash with Secretary	4	4	0			
(b) P. O. Savings Bank ...	3,983	13	9			
(c) Bengal Central Bank Ltd. ...	1,510	2	6			
(d) K. K. G. P. Fund ...	1,891	11	0			
(e) Cash Certificates (Face Value Rs. 2,000) ...	2,000	0	0	9,389	15	3
2. Subscriptions ...	684	12	0			
3. Admission Fee ...	20	0	0			
4. Sale Proceeds ...	284	1	0	988	13	0
5. Interest A/c :—						
(a) P. O. Savings Bank ...	55	15	11			
(b) Bengal Central Bank Ltd. ...	4	11	0			
(c) K. K. G. P. Fund ...	51	2	0	111	12	11
TOTAL ...				Rs. 10,490	9	2

DISBURSEMENTS

	Rs.	A.	P.	Rs.	A.	P.	Rs.	A.	P.
1. Establishment ...				685	0	0			
2. Meetings ..				42	13	6			
3. Books and Journals (including charges for book-binding) ...				194	0	6			
4. Bulletin :—									
(a) Blocks, etc. ...	61	3	0						
(b) Postage ...	70	1	3	131	4	3			
5. Printing and Stationery ...				67	9	6			
6. Postage ...				68	6	3			
7. Bank charges ...				5	6	0			
8. Miscellaneous (including conveyance and despatch of Library Books to Berhampore)				93	13	0	1,288	5	0
Closing Balance :—									
(a) Cash with Secretary					3	6			
(b) Balance at Bank :—									
(i) Imperial Bank of India	536	4	6						
(ii) Do. (K. K. G. P. Fund) ...	5	5	3						
(iii) Bengal Central Bank Ltd.	519	6	0						
(iv) P. O. Savings Bank	539	13	8	1,600	13	5			
(c) G. P. Notes (General Fund) (Face Value Rs. 6,000) ...				5,663	11	6			
(d) G. P. Notes (K. K. G. P. Fund) (Face Value Rs. 2,000) ...				1,937	7	9	9,202	4	2
TOTAL							Rs. 10,490	9	2

REPORT ON THE ACCOUNTS OF THE CALCUTTA MATHEMATICAL SOCIETY FOR THE YEAR ENDING 31ST DECEMBER, 1942

We have examined the accounts of the Calcutta Mathematical Society for the year ending 31st December, 1942. The accounts have been properly kept and supported, as far as possible, by vouchers. The financial condition of the Society is quite satisfactory.

K. N. CHAKRAVARTI
B. C. CHATTERJEE
Auditors.

CALCUTTA * UNIVERSITY

BULLETIN

OF THE

CALCUTTA MATHEMATICAL SOCIETY

A limited number of back Volumes of the Bulletin of the Calcutta Mathematical Society will be sold at reduced prices.

For particulars write to:—

THE SECRETARY, CALCUTTA MATHEMATICAL SOCIETY,
92, Upper Circular Road, Calcutta.

Printed and published by Bhupendralal Banerjee
The Calcutta University Press, 48, Hazra Road, Calcutta.

BULLETIN

OF THE

CALCUTTA

MATHEMATICAL SOCIETY

BOARD OF EDITORS

N. M. Basu, D.Sc. F. W. Levi, Dr.Phil.Nat. C. V. H. Rao, M.A.
N. R. Sen, D.Sc., Ph.D. M. R. Siddiqi, M.A., Ph.D.
S. Ghosh, D.Sc. (*Secretary*)

Vol. 35 June, 1943 No. 2

CONTENTS

Page

5. Some Infinite Integrals involving Bessel Functions of Imaginary Arguments—By Miss S. Sinha 37
6. On Mutually Self-polar Simplexes—By Asghar Hameed ... 43
7. Spontaneous Energy Function and Quantum Statics—By B. N. Biswas 55
8. Stress Systems in rotating Aeolotropic Discs—By S. Ghosh 61
9. Some Self-reciprocal Functions—By H. C. Gupta ... 67
10. On Harmonic Functions—By H. Sircar 71
11. On a Generalisation of Reidemeister's Figure in a Web—By A. C. Chowdhury 77
12. On Expansion in Eigen-functions of Boundary Value Problems VI—By S. Minakshisundaram 81

CALCUTTA UNIVERSITY PRESS
1943

All correspondence with the Society, subscriptions to the Bulletin admission fees and annual contributions of members are to be sent to the *Secretary, Calcutta Mathematical Society, 92, Upper Circular Road, Calcutta.*

Papers intended for publication in the Bulletin of the Society, and all Editorial Correspondence, should be addressed to the *Editorial Secretary, Calcutta Mathematical Society.*

The publications of the Calcutta Mathematical Society may be purchased direct from the Society's office, or from its agents—Messrs. Bowes & Bowes, Booksellers and Publishers, 1, Trinity Street, Cambridge, England.

NOTICE TO AUTHORS

The manuscript of each paper communicated for publication in this Journal should be legibly written (preferably type-written) on one side of the paper and should be accompanied by a short *abstract* of the paper.

References to literature in the text should be given, whenever possible, in chronological order, only the names of the authors and years of publication, in brackets, being given. They should be cited in full at the end of the paper, the author's name following in alphabetical order, and each reference is to be arranged in the following form, *viz.*, name or names of authors ; year of publication ; name of the journal (abbreviation) ; number of volume ; and lastly, the page number. The following would be a useful illustration :

Wilson, B. M., 1922, *Proc. Lond. Math. Soc.* (2), 21, 235.

Authors of papers printed in the Bulletin are entitled to receive, free of cost, 25 separate copies of their communications. They can, however, by previous notice to the Secretary, ascertain whether it will be possible to obtain more copies even on payment of the usual charges.

SOME INFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS OF IMAGINARY ARGUMENT

By

MISS S. SINHA

(Communicated by Dr. B. Mohan—Received November 2, 1942)

Consider the integral

$$I=\int_0^\infty x^{\rho-1}K_\lambda(ax)K_\mu(ax)\,{}_pF_q(a_1,\,a_2,\ldots a_p;\,b_1,\,b_2,\ldots b_q;\,bx^2)dx$$

where $K_m(x)$ is a Bessel function of imaginary argument and ${}_pF_q(a_1, a_2,\ldots a_p\,;\, b_1, b_2,\ldots b_q;\, bx^2)$ is the generalised hypergeometric function.

We are going to use the following result given by Titchmarsh (1937, 8.19.6)

$$\int_0^\infty x^{\rho-1}K_\lambda(ax)K_\mu(ax)dx$$

$$=\frac{\sqrt{\pi}}{4a^\rho}\cdot\frac{\Gamma\left(\frac{\rho+\lambda+\mu}{2}\right)\Gamma\left(\frac{\rho+\lambda-\mu}{2}\right)\Gamma\left(\frac{\rho-\lambda+\mu}{2}\right)\Gamma\left(\frac{\rho-\lambda-\mu}{2}\right)}{\Gamma\left(\frac{\rho}{2}\right)\Gamma\left(\frac{\rho+1}{2}\right)}\ldots(a)$$

where $R\,(\rho\pm\lambda\pm\mu)>0$.

We have

$$I=\int_0^\infty x^{\rho-1}K_\lambda(ax)K_\mu(ax)\sum_{r=0}^\infty\frac{\prod\limits_{s=1}^{s=p}(a_s,\,r)b^r}{\prod\limits_{t=1}^{t=q}(a_t,\,r)r\,!}x^{2r}dx$$

$$=\sum_{r=0}^\infty\frac{\prod\limits_{s=1}^{s=p}(a_s,\,r)b^r}{\prod\limits_{t=1}^{t=q}(a_t,\,r)r\,!}\int_0^\infty x^{\rho+2r-1}K_\lambda(ax)K_\mu(ax)dx$$

Here, term by term integration is justifiable, provided that $R\ (\rho\pm\lambda\pm\mu)>0$ and $q-p\geq1$.

Thus, on using (a) and the Duplication formula for Gamma function, we get

$$I=\frac{2^{\rho-3}}{a^\rho\Gamma(\rho)}\Gamma\left(\frac{\rho+\lambda+\mu}{2}\right)\Gamma\left(\frac{\rho+\lambda-\mu}{2}\right)\Gamma\left(\frac{\rho-\lambda+\mu}{2}\right)\Gamma\left(\frac{\rho-\lambda-\mu}{2}\right)$$

$$\times\,{}_{p+4}F_{q+2}\left\{\begin{matrix}\frac{\rho+\lambda+\mu}{2},\ \frac{\rho+\lambda-\mu}{2},\ \frac{\rho-\lambda+\mu}{2},\ \frac{\rho-\lambda-\mu}{2},\ a_1,\,a_2,\ldots a_p; & \\ & \frac{b}{a^2}\\ \frac{\rho}{2},\ \frac{\rho+1}{2},\ b_1,\,b_2,\ldots b_q; & \end{matrix}\right\}$$

where $q-p\geq1$ and $R(\rho\pm\lambda\pm\mu)\geq0$. ... I

Particular cases

1. Put $b=\frac{1}{4}$, $p=2$, $q=3$, $a_1=\alpha$, $a_2=\beta-\alpha$, $b_1=\beta$, $b_2=\frac{\beta}{2}$ and $b_3=\frac{\beta+1}{2}$.

Then I takes the form

$$\int_0^\infty x^{\rho-1}K_\lambda(ax)K_\mu(ax)\,{}_2F_3(\alpha,\beta-\alpha;\beta,\frac{\beta}{2},\frac{\beta+1}{2};\tfrac{1}{4}x^2)dx.$$

Using (Bailey, p. 97)

$$\begin{aligned}{}_2F_3(\alpha,\beta-\alpha;\beta,\frac{\beta}{2},\frac{\beta+1}{2};\tfrac{1}{4}x^2)&={}_1F_1(\alpha;\beta;x)\,{}_1F_1(\alpha;\beta;-x)\\&=e^x\,{}_1F_1(\alpha;\beta;x)\,{}_1F_1(\beta-\alpha;\beta;x),\end{aligned}$$

we get
$$\int_0^\infty x^{\rho-1}e^xK_\lambda(ax)K_\mu(ax)\,{}_1F_1(\alpha;\beta;x)\,{}_1F_1(\beta-\alpha;\beta;x)dx$$

$$=\frac{2^{\rho-3}}{a^\rho\,\Gamma(\rho)}\Gamma\left(\frac{\rho+\lambda+\mu}{2}\right)\Gamma\left(\frac{\rho+\lambda-\mu}{2}\right)\Gamma\left(\frac{\rho-\lambda+\mu}{2}\right)\Gamma\left(\frac{\rho-\lambda-\mu}{2}\right)$$

$$\times\,{}_6F_5\left\{\begin{matrix}\frac{\rho+\lambda+\mu}{2},\frac{\rho+\lambda-\mu}{2},\frac{\rho-\lambda+\mu}{2},\frac{\rho-\lambda-\mu}{2},\alpha,\beta-\alpha;\\ \\ \frac{\rho}{2},\frac{\rho+1}{2},\beta,\frac{\beta}{2},\frac{\beta+1}{2};\end{matrix}\quad\frac{1}{4a^2}\right\}$$

$$R(\rho\pm\lambda\pm\mu)>0.$$

(*a*) Since (MacRoberts, 1938)

$${}_1F_1(\tfrac{1}{2}+n;1+2n;x)=\Gamma(n+1)x^{-n}2^{2n}e^{\frac{x}{2}}I_n(\tfrac{x}{2})$$

provided that $R(n+\frac{1}{2})>0$, we get, on putting $\beta=2\alpha=2n+1$,

$$\int_0^\infty x^{\rho-1}e^{2x}K_\lambda(ax)K_\mu(ax)\{I_n(\tfrac{x}{2})\}^2dx$$

$$=\frac{2^{\rho-2n-3}\Gamma\left(n+\frac{\rho+\lambda+\mu}{2}\right)\Gamma\left(n+\frac{\rho+\lambda-\mu}{2}\right)\Gamma\left(n+\frac{\rho-\lambda+\mu}{2}\right)\Gamma\left(n+\frac{\rho-\lambda-\mu}{2}\right)}{a^{\rho+2n}\Gamma(2n+\rho)\{\Gamma(n+1)\}^2}$$

$$\times\,{}_5F_4\left\{\begin{matrix}n+\frac{\rho+\lambda+\mu}{2},n+\frac{\rho+\lambda-\mu}{2},n+\frac{\rho-\lambda+\mu}{2},n+\frac{\rho-\lambda-\mu}{2},n+\frac{1}{2};\\ \\ n+\frac{\rho}{2},n+\frac{\rho+1}{2},2n+1,n+1;\end{matrix}\quad\frac{1}{4a^2}\right\}$$

where $R(\rho+2n\pm\lambda\pm\mu)>0$, $a>\frac{3}{2}$, $R(n+\frac{1}{2})>0$ and $\rho>3$.

(*b*) Since

$$M_{k,m}(x)=x^{m+\frac{1}{2}}e^{-\frac{1}{2}x}\,{}_1F_1(m+\tfrac{1}{2}-k;2m+1;x),$$

by putting $\alpha = m-k+\frac{1}{2}$, $\beta = 2m+1$, $b=1$ and replacing ρ by $\rho+2m+1$, we get

$$\int_0^\infty x^{\rho-1} e^{2x} K_\lambda(ax) K_\mu(ax) M_{k,m}(x) M_{-k,m}(x)\,dx$$

$$= \frac{2^{\rho+2m-2}\,\Gamma\left(m+\frac{\rho+\lambda+\mu+1}{2}\right)\Gamma\left(m+\frac{\rho+\lambda-\mu+1}{2}\right)}{a^{\rho+2m+1}\,\Gamma(\rho+2m+1)}$$

$$\times\Gamma\left(m+\frac{\rho-\lambda+\mu+1}{2}\right)\Gamma\left(m+\frac{\rho-\lambda-\mu+1}{2}\right)$$

$$\times {}_6F_5\left\{\begin{matrix} m+\frac{\rho+\lambda+\mu+1}{2},\ m+\frac{\rho+\lambda-\mu+1}{2},\ m+\frac{\rho-\lambda+\mu+1}{2},\ m+\frac{\rho-\lambda-\mu+1}{2}, \\ m-k+\frac{1}{2},\ m+k+\frac{1}{2};\ m+\frac{\rho+1}{2},\ m+1+\frac{\rho}{2},\ 2m+1,\ m+1,\ m+\frac{1}{2}; \end{matrix}\ \frac{1}{4a^2}\right\}$$

where $R(\rho+m\pm\lambda\pm\mu) > -1$, $a > \frac{3}{2}$ and $\rho > 2$.

(c) Since

$$L_n^\alpha(x) = \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+1)}.\ {}_1F_1(-n;\ \alpha+1;\ x),$$

where $L_n^\alpha(x)$ is the generalized Laguerre polynomial of order n and n is a positive integer, we get by putting $\alpha=-n$ and $\beta=\alpha+1$,

$$\int_0^\infty x^{\rho-1} e^{\frac{1}{2}x} K_\lambda(ax) K_\mu(ax) L_n^\alpha(x) M_{n+\frac{\alpha+1}{2},\ \frac{\alpha}{2}}(x)\,dx$$

$$= \frac{2^{\rho+\frac{\alpha-5}{2}}\,\Gamma(\alpha+n+1)\,\Gamma\left(\frac{\alpha+1+2\rho+2\lambda+2\mu}{4}\right)\Gamma\left(\frac{\alpha+1+2\rho-2\lambda+2\mu}{4}\right)}{a^{\rho+\frac{\alpha+1}{2}}\,n!\,\Gamma(\alpha+1)\,\Gamma\left(\rho+\frac{\alpha+1}{2}\right)}$$

$$\times\Gamma\left(\frac{\alpha+1+2\rho+2\lambda-2\mu}{4}\right)\Gamma\left(\frac{\alpha+1+2\rho-2\lambda-2\mu}{4}\right)$$

$$\times {}_6F_5\left\{\begin{matrix} \frac{\alpha+1+2\rho+2\lambda+2\mu}{4},\ \frac{\alpha+1+2\rho-2\lambda+2\mu}{4},\ \frac{\alpha+1+2\rho+2\lambda-2\mu}{4}, \\ \frac{\alpha+1+2\rho-2\lambda-2\mu}{4},\ -n,\ n+\alpha+1; \\ \frac{2\rho+\alpha+1}{4},\ \frac{2\rho+\alpha+3}{4},\ \alpha+1,\ \frac{\alpha+1}{2},\ \frac{\alpha}{2}+1; \end{matrix}\ \frac{1}{4a^2}\right\}$$

where $R\left(\frac{\alpha}{2}+\rho\pm\lambda\pm\mu\right) > -\frac{1}{2}$, $a > 1$ and n is a positive integer.

Similarly, we can obtain from (1) other interesting results involving integrals of products of Sonine polynomial, Bateman's function, Weber's parabolic cylinder function and Hermite polynomial in different combinations.

2(a) Put $a_1=\dfrac{1+\nu+\delta}{2}$, $a_2=\dfrac{2+\nu+\delta}{2}$, $b_1=1+\nu$, $b_2=1+\delta$, $p=2$

$b_3=1+\nu+\delta$, $q=3$, $\rho=p+\nu+\delta$ and $b=-1$ in (I).

Since (Watson, 1922, p. 147)

$$ {}_2F_3\left\{\frac{1+\nu+\delta}{2},\ \frac{2+\nu+\delta}{2}\ ;\ 1+\nu,\ 1+\delta,\ 1+\nu+\delta;\ -x^2\right\} $$

$$ =\Gamma(\nu+1)\Gamma(\delta+1)\left(\frac{x}{2}\right)^{-\nu-\delta} J_\nu(x) J_\delta(x), $$

The integral becomes

$$ \int_0^\infty x^{p-1} K_\lambda(ax) K_\mu(ax) J_\nu(x)\, J_\delta(x)\, dx $$

$$ =\frac{2^{p-3}\Gamma\left(\dfrac{p+\nu+\delta+\lambda+\mu}{2}\right)\Gamma\left(\dfrac{p+\nu+\delta+\lambda-\mu}{2}\right)}{a^{p+\nu+\delta}\Gamma(\nu+1)\Gamma(\delta+1)\Gamma(p+\nu+\delta)} $$

$$ \times\Gamma\left(\frac{p+\nu+\delta-\lambda+\mu}{2}\right)\Gamma\left(\frac{p+\nu+\delta-\lambda-\mu}{2}\right) $$

$$ \times\, {}_6F_5\left\{\begin{matrix} \dfrac{p+\nu+\delta+\lambda+\mu}{2}, \dfrac{p+\nu+\delta+\lambda-\mu}{2}, \dfrac{p+\nu+\delta-\lambda+\mu}{2}, \dfrac{p+\nu+\delta-\lambda-\mu}{2}, \\ \dfrac{1+\nu+\delta}{2}, \dfrac{2+\nu+\delta}{2}; \\ p+\nu+\delta,\ \dfrac{p+\nu+\delta+1}{2},\quad \nu+1,\quad \delta+1,\quad 1+\nu+\delta; \end{matrix}\ -\frac{1}{a^2}\right\} $$

where $\qquad R(p+\nu+\delta\pm\lambda\pm\mu)>0,\ a>0$ and $p>2$.

(b) Put $a_1=\dfrac{1+\nu+\delta}{2}$, $a_2=\dfrac{2+\nu+\delta}{2}$, $b_1=1+\nu$, $b_2=1+\delta$, $b_3=1+\nu+\delta$,

$p=2$, $q=3$, $l=p+\nu+\delta$ and $b=1$.

Now, we know that (Meijer, 1935, § 7)

$$ {}_2F_3\left\{\frac{1+\nu+\delta}{2},\ \frac{2+\nu+\delta}{2}\ ;\ 1+\nu,\ 1+\nu+\delta,\ 1+\delta;\ x^2\right\} $$

$$ =\Gamma(\nu+1)\Gamma(\delta+1)\left(\frac{x}{2}\right)^{-\nu-\delta} I_\nu(x) I_\delta(x). $$

Therefore $\int_0^\infty x^{p-1}\mathrm{K}_\lambda(ax)\mathrm{K}_\mu(ax)\mathrm{I}_\nu(x)\mathrm{I}_\delta(x)dx$

$$=\frac{2^{p-3}\Gamma\left(\frac{p+\nu+\delta+\lambda+\mu}{2}\right)\Gamma\left(\frac{p+\nu+\delta+\lambda-\mu}{2}\right)\Gamma\left(\frac{p+\nu+\delta-\lambda+\mu}{2}\right)}{a^{p+\nu+\delta}\Gamma(\nu+1)\Gamma(\delta+1)\Gamma(p+\nu+\delta)}$$

$$\times\Gamma\left(\frac{p+\nu+\delta-\lambda-\mu}{2}\right)$$

$$\times{}_6\mathrm{F}_5\left\{\begin{matrix}\frac{p+\nu+\delta+\lambda+\mu}{2},\ \frac{p+\nu+\delta+\lambda-\mu}{2},\ \frac{p+\nu+\delta-\lambda+\mu}{2},\ \frac{p+\nu+\delta-\lambda-\mu}{2}, \\ \frac{1+\nu+\delta}{2},\ \frac{2+\nu+\delta}{2}; \\ p+\delta+\nu,\ \frac{p+\nu+\delta+1}{2},\ 1+\nu,\ 1+\delta,\ 1+\nu+\delta;\end{matrix}\ \frac{1}{a^2}\right\}$$

where $\mathrm{R}(p+\nu+\delta\pm\lambda\pm\mu)>0$, $a>1$ and $p \geq 3$.

3. Putting $p=1$, $q=2$, $b=-\frac{c^2}{4}$; $a_1=1$, $b_1=\frac{3}{2}$, $b_2=\nu+\frac{3}{2}$ and replacing ρ by $\rho+\nu+1$, we get a formula involving Struve's functions which has been proved by Mohan (1942).

4. Put $p=0$, $q=3$, $b=-\frac{1}{6^4}$, $b_1=\frac{1}{2}+p$, $b_2=1+p$, $b_3=1+2p$ and $\rho=l+2p$ in (I).

Since (Meijer, 1935, p. 11)

$${}_0\mathrm{F}_3\left(\tfrac{1}{2}+p,\ 1+p,\ 2p+1;\ -\frac{x^2}{6^4}\right)\frac{x^{2p}}{\{\Gamma(1-2p)\}^2 2^{4p}}=\mathrm{I}_{2p}(\sqrt{x})\mathrm{J}_{2p}(\sqrt{x})$$

we have $\int_0^\infty x^{\rho-1}\mathrm{K}_\lambda(ax)\mathrm{K}_\mu(ax)\mathrm{I}_{2p}(\sqrt{x})\mathrm{J}_{2p}(\sqrt{x})dx$

$$=\frac{\Gamma\left(p+\frac{\rho+\lambda+\mu}{2}\right)\Gamma\left(p+\frac{\rho-\lambda+\mu}{2}\right)\Gamma\left(p+\frac{\rho+\lambda-\mu}{2}\right)\Gamma\left(p+\frac{\rho-\lambda-\mu}{2}\right)}{2^{2p-l+3}a^{2p+l}\Gamma(2p+\rho)\{\Gamma(1-2p)\}^2}$$

$$\times{}_4\mathrm{F}_5\left\{\begin{matrix}p+\frac{\rho+\lambda+\mu}{2},\ p+\frac{\rho-\lambda+\mu}{2},\ p+\frac{\rho+\lambda-\mu}{2},\ p+\frac{\rho-\lambda-\mu}{2}; \\ 2p+\rho,\ p+\frac{\rho}{2},\ p+1,\ p+\frac{1}{2},\ 2p+1;\end{matrix}\ -\frac{1}{6^4a^2}\right\},$$

where $\mathrm{R}(\rho+2p\pm\lambda\pm\mu)>0$, $l>\frac{5}{2}$ and $a>0$.

5. Put $p=0$, $q=1$, $b_1=n+1$, $b=-\frac{c^2}{4}$ and $\rho=l+n$ in (1).

Since $$J_n(x)=\frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n {}_0F_1(1+n;\ -\tfrac{1}{4}x^2),$$

we have

$$\int_0^\infty x^{\rho-1}K_\lambda(ax)\,K_\mu(ax)\,J_n(cx)\,dx$$

$$=\frac{c^n.\,2^{\rho-3}\Gamma\left(\frac{\rho+n+\lambda+\mu}{2}\right)\Gamma\left(\frac{\rho+n-\lambda+\mu}{2}\right)\times\Gamma\left(\frac{\rho+n+\lambda-\mu}{2}\right)\Gamma\left(\frac{\rho+n-\lambda-\mu}{2}\right)}{a^{\rho+n}\Gamma(n+1)\Gamma(\rho+n)}$$

$$\times{}_4F_3\left\{\begin{matrix}\frac{\rho+n+\lambda+\mu}{2},\ \frac{\rho+n-\lambda+\mu}{2},\ \frac{\rho+n+\lambda-\mu}{2},\ \frac{\rho+n-\lambda-\mu}{2}; \\ \frac{\rho+n}{2},\ \frac{\rho+n+1}{2},\ n+1;\end{matrix}\ -\frac{c^2}{4a^2}\right\}$$

where $R(\rho+n\pm\lambda\pm\mu)>0$, $\rho>\frac{5}{2}$ and $a>0$.

Proceeding exactly as in the beginning of this note, we can show that

$$\int_0^\infty x^{\rho-1}K_\lambda(ax)K_\mu(ax)I_\nu(bx)\,dx$$

$$=\frac{2^{\rho-3}b^\nu\Gamma\left(\frac{\rho+\nu+\lambda+\mu}{2}\right)\Gamma\left(\frac{\rho+\nu-\lambda+\mu}{2}\right)}{a^{\rho+\nu}\ \Gamma(\nu+1)\Gamma(\rho+\nu)}\times\Gamma\left(\frac{\rho+\nu-\lambda-\mu}{2}\right)\Gamma\left(\frac{\rho+\nu+\lambda-\mu}{2}\right)$$

$$\times{}_4F_3\left\{\begin{matrix}\frac{\rho+\nu+\lambda+\mu}{2},\ \frac{\rho+\nu-\lambda+\mu}{2},\ \frac{\rho+\nu+\lambda-\mu}{2},\ \frac{\rho+\nu-\lambda-\mu}{2}; \\ \nu+1,\ \frac{\rho+\nu}{2},\ \frac{\rho+\nu+1}{2};\end{matrix}\ \frac{b^2}{4a^2}\right\}$$

where $R(\rho+\nu\pm\lambda\pm\mu)>0$, $a>\frac{b}{2}$ and $l>\frac{5}{2}$.

I am thankful to Dr. B. Mohan for his kind help in the preparation of this paper.

HINDU UNIVERSITY,
BENARES.

References

Bailey, W. N., *Generalised Hypergeometric Series* (Cambridge).
MacRoberts, T. M., 1938, *Phil. Mag.*, 26, 82-93.
Meijer, C. S., 1935, *Proc. Lond. Math. Soc.*, 40, 1-22.
Mohan, B., 1942, *Bull. Cal. Math. Soc.*, 34, 55-59.
Titchmarsh, E. C., 1937, *Theory of Fourier Integrals* (Oxford).
Watson, G. N., 1922, *Theory of Bessel Functions* (Cambridge).

ON MUTUALLY SELF-POLAR SIMPLEXES

By

Asghar Hameed

(Communicated by Prof. C. V. H. Rao—Received December 21. 1942)

Introduction. This paper is a generalization to space of n dimensions with particular reference to space of four dimensions, of the properties of mutually Self-Polar Tetrahedra discussed in my previous paper (1941).

If the co-ordinates of a point P referred to an arbitrary simplex $A_1A_2 \ldots A_{n+1}$, in space of n dimensions, be $(x'_1, x'_2, \ldots x'_{n+1})$, we define the prime $\sum_{r=1}^{n+1} \frac{x_r}{x'_r} = 0$ to be the *Polar Prime* of P with respect to the simplex $A_1A_2 \ldots A_{n+1}$, and call P the *pole* of that prime with respect to the same simplex.

A simplex S_1 will be said to be *self-polar* with respect to another simplex S_2, if the polar prime with respect to S_2, of any vertex of S_1, is its opposite prime face. We show that pairs of mutually self-polar simplexes exist in space of any number of dimensions (§ 4).

In § 7, we prove that if S_1 be a simplex self-polar with respect to another simplex S_2, and S'_1 and S'_2 be their respective reciprocal simplexes in regard to any quadric variety, then S'_1 is also self-polar with respect to S'_2. We define inversion with respect to spaces $A_1A_2 \ldots A_r$ and $A_{r+1}A_{r+2} \ldots A_{n+1}$, and prove that a point and its polar prime with respect to a simplex invert respectively into a point and its polar prime with respect to the inverse simplex (§§ 8, 9). In some of the following articles, there is a discussion on two particular types of pairs of mutually self-polar simplexes, dealing with their reciprocities, perspectivities, and the cross-ratios arising therefrom

Considering the problem particularly in space of four dimensions, we arrive in space of three dimensions, at a quartic surface, believed to be new, having ten double points, which arrange themselves according to the three-dimensional 10_3 configuration of Desargue (§ 18). We give a particular type of a pair of mutually self-polar simplexes such that the five lines of an associated set can always be thought of as the joins of their corresponding vertices (§ 19).

§ 1. We start with an arbitrary simplex $A_1A_2 \ldots A_{n+1}$ and a point P of general position in the space of n dimensions fixed by the simplex. Regarding $A_1A_2 \ldots A_{n+1}$ as the simplex of reference and P as the unit point, the polar prime of P *w.r.t.* the simplex $A_1A_2 \ldots A_{n+1}$ has the algebraic equation $\Sigma x_1 = 0$. We proceed to give the following geometrical interpretations of this algebraic definition:—

(1) Let the prime P $A_1A_2 \ldots A_{n-1}$ meet the line A_nA_{n+1} in the point $Q_{n,n+1}$ whose harmonic conjugate *w.r.t.* A_n and A_{n+1} is say, $Q'_{n,n+1}$. Then ${}^{n+1}C_2$

points like $Q'_{n,n+1}$ *lie in a prime which is the polar prime of* P *w.r.t. the simplex* $A_1A_2\ldots A_{n+1}$.

(2) Let the space P $A_1A_2\ldots A_{n-r-1}$ meet the space $A_{n-r}\ A_{n-r+1}\ldots A_{n+1}$ in the point Q. This point has a polar space of r dimensions *w.r.t.* the simplex $A_{n-r}A_{n-r+1}\ldots A_{n+1}$. *Such* ${}^{n+1}C_{r+2}$ *polar spaces lie in a prime which is the polar prime of* P *w.r.t. the simplex* $A_1A_2\ldots A_{n+1}$. And this is true for $r=1$, $2, 3,\ldots n-2$.

(3) Let PA_1 meet the prime $A_2A_3\ldots A_{n+1}$ in the point A'_1, similar interpretations being given for the points A'_2, $A'_3\ldots.A'_{n+1}$. Then *the* $n+1$ *spaces, such as is given by the intersection of the primes* $A_1A_2\ldots A_n$ and $A'_1A'_2\ldots A'_n$, *lie in a prime which is the polar prime of* P *w.r.t. the simplex* $A_1A_2\ldots A_{n+1}$. In fact these $n+1$ spaces are the same as the polar spaces of the previous case when $r=n-2$.

(4) When $n+1=2r$, the simplex of reference may be written as $A_1A_2\ldots A_rA_{r+1}\ldots A_{2r}$. We can draw a unique transversal from P to the two spaces $A_1A_2\ldots A_r$ and $A_{r+1}A_{r+2}\ldots A_{2r}$. Let this transversal meet the two spaces respectively in the points $Q_{1,2,\ldots r}$ and $Q_{r+1,r+2\ldots 2r}$. Let the harmonic conjugate of P *w.r.t.* these two points be called $P_{1,2\ldots r}(\equiv P_{r+1,r+2\ldots 2r})$. Corresponding to the $\frac{1}{2}\,{}^{2r}C_r$ ways of dividing $2r$ things in two equal parts, we get $\frac{1}{2}\,{}^{2r}C_r$ points of the type $P_{1,2\ldots r}$. *These* $\frac{1}{2}\,{}^{2r}C_r$ *points lie in a prime which is the polar prime of* P *w.r.t. the simplex* $A_1A_2\ldots A_{n+1}$.

§ 2. Now let any simplex be denoted by its prime faces a_1, $a_2,\ldots a_{n+1}$ and let p be a prime of general position in the space of n dimensions fixed by the simplex, which we take as the simplex of reference. Let p be the unit prime $\Sigma x_r=0$. Then the pole of p *w.r.t.* the simplex $a_1a_2\ldots a_{n+1}$ has co-ordinates $(1, 1,\ldots 1)$. We now give the following geometrical constructions:—

(1) There is a unique prime $q_{n,n+1}$ which passes through the point $pa_1a_2\ldots a_{n-1}$ (*i.e*, the point of intersection of the n primes p, a_1, $a_2,\ldots a_{n-1}$) and through the intersection of the primes a_n and a_{n+1}. Let $q'_{n,n+1}$ be the harmonic conjugate of $q_{n,n+1}$ *w. r. t.* the primes a_n and a_{n+1}. Then the ${}^{n+1}C_2$ *primes of the type* $q'_{n,n+1}$ *pass through a common point which is the pole of* p *w.r.t. the simplex* $a_1a_2\ldots a_{n+1}$.

(2) There is a unique prime which passes through the space of intersection of the primes p and a_1 and through the point of intersection of the primes a_2, $a_3\ldots$ and a_{n+1}. Let this prime be called a'_1. In this way we get $n+1$ primes a'_1, $a'_2\ldots a'_{n+1}$. Then $n+1$ *lines of the type given by the join of the points* $a_1a_2\ldots a_n$ and $a'_1a'_2\ldots a'_n$, *all pass through a common point which is the pole of* p *w.r.t. the simplex* $a_1a_2\ldots a_{n+1}$.

(3) When $n+1=2r$, the simplex may be written as $a_1a_2\ldots a_ra_{r+1}\ldots a_{2r}$. There is a unique space of $n-2$ dimensions which lies in the prime p and which, together with each of the two spaces $a_1\ a_2\ldots a_r$ and $a_{r+1}\ a_{r+2}\ldots a_{2r}$ determines a prime. Let the harmonic conjugate of p *w.r.t.* these two primes be called $p_{1,2,\ldots r}\equiv(p_{r+1,\ r+2,\ldots 2r})$. We get $\frac{1}{2}\,{}^{2r}C_r$ primes of the type $p_{1,2\ldots r}$, *all of*

which pass through a common point which is the pole of p w.r.t. the simplex $a_1a_2\ldots a_{n+1}$.

§ 3. The polar prime of p *w.r.t.* the simplex $A_1A_2\ldots A_{n+1}$ as already defined is the polar prime of P *w.r.t.* the degenerate point-variety consisting of the $n+1$ prime faces of the simplex, and also *w. r. t.* the degenerate class variety consisting of the $n+1$ vertices of the simplex.

We may remark that if the point P lies on a single prime face of the simplex $A_1A_2\ldots A_{n+1}$, then its polar prime *w.r.t.* it is that prime face itself. If, however, P lies on the space of intersection of two or more prime faces of the simplex, then its polar prime *w.r.t.* it becomes indeterminate. Throughout this paper we shall assume that P does not lie on any prime face of the simplex *w.r.t.* which its polar prime is to be considered.

§ 4. Let the co-ordinates of the vertices of a simplex $A'_1\,A'_2\ldots A'_{n+1}$ referred to a simplex $A_1\,A_2\ldots A_{n+1}$, both lying in the same space of n dimensions, be given respectively by the rows of the following determinant of $(n+1)$th order:

$$\begin{vmatrix} k & 1 & . & . & . & 1 & 1 \\ 1 & k & . & . & . & 1 & 1 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & 1 & . & . & . & k & 1 \\ 1 & 1 & . & . & . & 1 & k \end{vmatrix}$$

in which each diagonal element is k, and every other element is unity. It is obvious that the simplex $A'_1A'_2\ldots A'_{n+1}$ will be self-polar *w.r.t.* the simplex $A_1\,A_2\ldots A_{n+1}$ if

$$k+\frac{1}{k}+n-1=0. \qquad \ldots \quad \text{(A)}$$

And when this condition is satisfied, the simplex of reference is likewise self-polar *w.r.t.* the simplex $A'_1A'_2\ldots A'_{n+1}$. For the co-ordinates of the vertices of the former referred to the latter, are given by the rows of the above determinant in which k is replaced by $\frac{1}{k}$, provided that the unit point is suitably chosen. This is clear from the transformation,

$$(\xi_1,\,\xi_2,\ldots\xi_{n+1}) = \begin{bmatrix} k' & 1 & . & . & . & 1 & 1 \\ 1 & k' & . & . & . & 1 & 1 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & 1 & . & . & . & k' & 1 \\ 1 & 1 & . & . & . & 1 & k' \end{bmatrix} (x_1,\,x_2,\ldots x_{n+1})$$

which, except for a multiplied constant, leads to a similar transformation expressing $x_1, x_2, \ldots x_{n+1}$ in terms of $\xi_1, \xi_2, \ldots \xi_{n+1}$, with k' replaced by k, where $kk'=1$. This proves that *pairs of mutually self-polar simplexes exist in space of any number of dimensions.*

The simplex $A'_1 A'_2 \ldots A'_{n+1}$ for each of the two values of k, is in perspective with the simplex of reference $A_1 A_2 \ldots A_{n+1}$, the centre of perspective being the unit point. On each of the $n+1$ rays of perspective, there lie the unit point, one vertex of the simplex of reference, and two vertices respectively of the two solutions for the simplex $A'_1 A'_2 \ldots A'_{n+1}$. *The cross-ratio of these four points is the same on all the $n+1$ rays of perspective.* When the points are taken in a certain order, the value of this cross-ratio is $(k_2-1)/(k_1-1)$, where k_1 and k_2 are the two roots of the equation (A).

It may be shown that the two simplexes $A_1 A_2 \ldots A_{n+1}$ and $A'_1 A'_2 \ldots A'_{n+1}$, for $n>3$, are reciprocal *w.r.t.* a quadric variety provided that the correspondence between their vertices in this reciprocity is such that, if the vertex A_r of the former simplex corresponds to the vertex A'_s of the latter simplex, then the vertex A_s of the former corresponds necessarily to the vertex A'_r of the latter.

§5. When $n>3$, we get $n+1$, pairs of simplexes of the type considered in § 4, having a given point as a common vertex, and being mutually self-polar *w.r.t.* a given simplex. For, taking the given point as the unit point, referred to the given simplex, the determinant of the previous article may now be written as

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & \ldots & \ldots & 1 & 1 \\ 1/k & k & 1 & \ldots & \ldots & \ldots & 1 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1/k & 1 & 1 & \ldots & \ldots & \ldots & k & 1 \\ 1/k & 1 & 1 & \ldots & \ldots & \ldots & 1 & k \end{vmatrix}$$

For each of the two values of k given by the equation (A) of § 4, the rows of the above determinant give the co-ordinates of the vertices of a simplex which is mutually self-polar *w.r.t.* the simplex of reference. This gives us one pair of simplexes with the stated property. Now, interchanging the first column of this determinant with each of the other columns in turn, we get in all $n+1$ determinants which give the required $n+1$ pairs. That this statement holds for $n>3$ may be proved by considering as to when the two simplexes of each pair, or the $n+1$ pairs themselves, are distinct.

§ 6. When $n+1$ is prime, let ρ be an imaginary root of the equation $x^{n+1}=1$. And let the rows of the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & \ldots & \ldots & 1 & 1 \\ 1 & \rho & \rho^2 & \ldots & \ldots & \ldots & \rho^{n-1} & \rho^n \\ 1 & \rho^2 & (\rho^2)^2 & \ldots & \ldots & \ldots & (\rho^2)^{n-1} & (\rho^2)^n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \rho^{n-1} & (\rho^{n-1})^2 & \ldots & \ldots & \ldots & (\rho^{n-1})^{n-1} & (\rho^{n-1})^n \\ 1 & \rho^n & (\rho^n)^2 & \ldots & \ldots & \ldots & (\rho^n)^{n-1} & (\rho^n)^n \end{vmatrix}$$

give the co-ordinates of the vertices of a simplex $A''_{11}\,A''_{12}\ldots A''_{1n+1}$ referred to the simplex $A_1A_2\ldots A_{n+1}$. The two simplexes are then mutually self-polar. This may also be proved in a manner similar to that of § 4.

Keeping the first column of the above determinant as it is, and interchanging the other columns between themselves in all possible ways, we get $\underline{|n}$ simplexes of the type $A''_{11}A''_{12}\ldots A''_{1n+1}$. When $n=2$, however, these $\underline{|n}$, *i.e.*, 2 simplexes coincide. Since all of these $\underline{|n}$ simplexes have the unit point as a common vertex, we arrive at the result that for $n>2$, there are $\underline{|n}$ simplexes of the type considered in this article, all of which have an arbitrary point as a common vertex and are mutually self-polar *w.r.t.* a given simplex.

§ 7. Considering case (1), § 1 and case (1), § 2, we can show that *pole and polar prime w.r.t. a simplex, reciprocate in regard to a quadric variety, respectively into polar prime and pole w.r.t. the reciprocal simplex.* From this follows the result that, if S_1 be a simplex self-polar *w.r.t.* another simplex S_2, and S'_1 and S'_2 be their respective reciprocal simplexes in regard to a quadric variety, then S'_1 is also self-polar *w.r t.* S'_2. In particular S_1 may be supposed to be self-conjugate *w.r.t.* the quadric variety, so that in the above enunciation, S'_1 may be replaced by S_1.

§ 8. We now come to the theory of inversion. Let A_1 be an arbitrary point and $A_2A_3\ldots A_{n+1}$ an arbitrary prime; both lying in the space of n dimensions. Let P be another point of the space. The line A_1P meets the prime $A_2A_3\ldots A_{n+1}$ in a point. Call it P_1. On the line A_1P, choose a point P' such that $(A_1PP_1P')=K$, an arbitrary constant. Then P' will be said to be the *inverse* of P *w.r.t.* the point A_1 and the prime $A_2A_3\ldots A_{n+1}$. In this process of inversion, points invert into points, lines into lines, etc., and cross-ratios remain unaltered.

Now, from case (1), § 1, it is clear that the figure of a point and its polar prime *w.r.t.* a simplex depends entirely on harmonic cross-ratios. Hence, *a point and its polar prime w.r.t. a simplex, invert respectively into a point and its polar prime w.r.t. the inverse simplex.* As a consequence of this it follows that, *if a simplex* S_1 *is self-polar w.r.t. a simplex* S_2, *then so is the inverse of* S_1 *self-polar w.r.t. the inverse of* S_2.

§ 9. We may similarly define an inversion *w.r.t.* two general spaces $A_1A_2\ldots A_r$ and $A_{r+1}A_{r+2}\ldots A_{n+1}$ $1\leq r\leq n$, lying in the space of n dimensions. Through an arbitrary point P, a unique transversal can be drawn to the two spaces. If Q_1 and Q_2 be the two points in which the transversal meets the two spaces respectively, and P' be the point on the transversal, such that $(Q_1PQ_2P')=K'$, an arbitrary constant, then P' will be said to be the *inverse* of P *w.r.t.* the two spaces $A_1A_2\ldots A_r$ and $A_{r+1}A_{r+2}\ldots A_{n+1}$. We remark that we have to consider the order in which the two spaces are mentioned. Thus the inverse of P *w.r.t.* the spaces $A_{r+1}A_{r+2}\ldots A_{n+1}$ and $A_1A_2\ldots A_r$ is not P'. It will be a point P'', such that $(Q_2PQ_1P'')=K'$. When, however, the inversion is harmonic, *i.e.*, when $K'=-1$, P'' will coincide with P'.

Such an inversion as is defined in the present article is a combination of r successive inversions of the previous article. More precisely, the inverse of P *w.r.t.* the two spaces $A_1A_2 \ldots A_r$ and $A_{r+1}A_{r+2} \ldots A_{n+1}$ may be obtained by applying inversion to P *w.r.t.* the point A_1 and the prime $A_2A_3 \ldots A_{n+1}$, followed by an inversion *w.r.t.* the point A_2 and the prime $A_1A_3A_4 \ldots A_{n+1}$, and so on, ultimately followed by an inversion *w.r.t.* the point A_r and the prime $A_1A_2 \ldots A_{r-1}A_{r+1}A_{r+2} \ldots A_{n+1}$, the constant of inversion being the same in all of these successive inversions.

Similarly, the present inversion is a combination of $n+1-r$ successive inversions of § 8, namely an inversion *w r.t.* the point A_{r+1} and the prime $A_1A_2 \ldots A_rA_{r+2}A_{r+3} \ldots A_{n+1}$, followed by an inversion *w.r.t.* the point A_{r+2} and the prime $A_1A_2 \ldots A_{r+1}A_{r+3}A_{r+4} \ldots A_{n+1}$, and so on, the constant of inversion in these successive inversions being, however, $1/K'$ instead of K'.

It is now clear that the remarks made in the last paragraph of § 8 hold also when the inversion is that as considered in the present article.

§ 10. Consider two simplexes $A'_{11}A'_{12} \ldots A'_{1n+1}$ and $A'_{21}A'_{22} \ldots A'_{2n+1}$ of the type discussed in § 5. The co-ordinates of their vertices are given below:

A'_{11}	$(1, \quad 1, \quad 1, \ldots, \quad 1, \quad 1)$	A'_{21}	$(x'_1, \quad x'_2, \quad x'_3, \ldots x'_n, \quad x'_{n+1})$
A'_{12}	$(1/k_1, k_1, 1, \ldots, \quad 1, \quad 1)$	A'_{22}	$(x'_1/k_2, k_2x'_2, x'_3, \ldots, x'_n, \quad x'_{n+1})$
...	... ...	...	... ...
...	... ...	...	... ...
A'_{1n}	$(1/k_1, \quad 1, \quad 1, \ldots, k_1, \quad 1)$	A'_{2n}	$(x'_1/k_2, \quad x'_2, \quad x'_3, \ldots, \quad k_2x'_n, \quad x'_{n+1})$
A'_{1n+1}	$(1/k_1, \quad 1, \quad 1, \ldots, \quad 1, \quad k_1)$	A'_{2n+1}	$(x'_1/k_2, \quad x'_2, \quad x'_3, \ldots, \quad x'_n, \quad k_2x'_{n+1})$

where k_1 and k_2 are, of course, the roots of equation (A) of § 4.

There exists a quadric variety, namely, $\sum\limits_{r=1}^{n+1} x_r^2/x'_r=0$, in regard to which the simplex of reference $A_1A_2 \ldots A_{n+1}$ is self-conjugate, and *w.r.t.* which the two simplexes $A'_{11}A'_{12} \ldots A'_{1n+1}$ and $A'_{21}A'_{22} \ldots A'_{2n+1}$ are reciprocal.

Further, for $n>3$, corresponding to the $n+1$ pairs of simplexes of § 5, we also get $n+1$ pairs of simplexes as the pair $A'_{11}A'_{12} \ldots A'_{1n+1}$ and $A'_{21}A'_{22} \ldots A'_{2n+1}$. The two simplexes of each of these pairs will respectively have $(1,1, \ldots 1)$ and $(x'_1, x'_2, \ldots, x'_{n+1})$ as one of their vertices, and will be reciprocal *w.r.t.* the same quadric variety.

Putting $x'_1=x'_2= \ldots =x'_{n+1}$, the two simplexes $A'_{11}A'_{12} \ldots A'_{1n+1}$ and $A'_{21}A'_{22} \ldots A'_{2n+1}$ belong to one of the $n+1$ pairs of § 5, and are reciprocal *w.r.t.* $\Sigma x_1^2=0$. In fact, the two simplexes of each of these pairs are reciprocal *w.r.t.* this quadric variety.

§ 11. When $n+1$ is prime, consider two simplexes $A''_{11}A''_{12} \ldots A''_{1n+1}$ and $A''_{21}A''_{22} \ldots A''_{2n+1}$ of the type mentioned in § 6, in relation to the simplex of reference $A_1A_2 \ldots A_{n+1}$. The co-ordinates of the vertices of $A''_{11}A''_{12} \ldots A''_{1n+1}$ are given in § 6, and those of the vertices of $A''_{21}A''_{22} \ldots A''_{2n+1}$ are given below:

$$\begin{array}{lllllll}
A''_{21} & \{x'_1, & x'_2, & x'_3, & \ldots\ldots\ldots & x'_n, & x'_{n+1}\} \\
A''_{22} & \{x'_1, & \rho x'_2, & \rho^2 x'_3, & \ldots\ldots\ldots & \rho^{n-1}x'_n, & \rho^n x'_{n+1}\} \\
\ldots & \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & & \ldots \\
A''_{2n} & \{x'_1, & \rho^{n-1}x'_2, & (\rho^{n-1})^2 x'_3, & \ldots\ldots\ldots & (\rho^{n-1})^{n-1}x'_n, & (\rho^{n-1})^n x'_{n+1}\} \\
A''_{2n+1} & \{x'_1, & \rho^n x'_2. & (\rho^n)^2 x'_3, & \ldots\ldots\ldots & (\rho^n)^{n-1}x'_n, & (\rho^n)^n x'_{n+1}\}
\end{array}$$

There exist $n+1$ quadric varieties, in regard to each of which the simplex $A_1A_2\ldots A_{n+1}$ is self-conjugate, and *w. r. t.* each of which the two simplexes $A''_{11}A''_{12}\ldots A''_{1n+1}$ and $A''_{21}A''_{22}\ldots A''_{2n+1}$ are reciprocal in some order. The equations of these $n+1$ quadric varieties may be put down as

$$\sum_{r=1}^{n+1} \{x_r^2/(\rho^s)^{r-1}x'_r\}=0, \; s=0, 1, 2,\ldots n.$$

§ 12. Let R_s denote the sum

$$x_1+\rho^{s-1}x_2+(\rho^{s-1})^2x_3+\ldots+(\rho^{s-1})^{n-1}x_n+(\rho^{s-1})^n x_{n+1},$$

where $1 \leq s \leq n+1$.

Then the simplexes $A_1A_2\ldots A_{n+1}$ and $A''_{11}A''_{12}\ldots A''_{1n+1}$ are reciprocal in some order, *w.r.t.* the quadric variety

$$\sum_{s=1}^{n+1} x_sR_s=0. \qquad \ldots (i)$$

For example, the reciprocal *w.r.t.* this variety, of the point A_s is the prime $R_s=0$, which is the prime face of the simplex $A''_{11}A''_{12}\ldots A''_{1n+1}$, opposite to the vertex A''_{1n+3-s}, where A''_{1n+2} is to be taken identical with A''_{11}.

Now, changing ρ into any other imaginary $(n+1)$th root of unity, we see that *w.r.t.* the quadric variety (*i*), the simplex $A_1A_2\ldots A_{n+1}$ still reciprocates into the simplex $A''_{11}A''_{12}\ldots A''_{1n+1}$ in some order. And in all of these n cases the reciprocal of the vertex A_1 is the prime face of the simplex $A''_{11}A''_{12}\ldots A''_{1n+1}$ oppsite to the vertex A''_{11}.

By linear transformations, the vertex A''_{1s} can be made the unit point, the simplex of reference remaining the same. And then it may be seen that the rows of the determinant of § 6, which give the co-ordinates of the vertices of the simplex $A''_{11}A''_{12}\ldots A''_{1n+1}$ get interchanged between themselves. We thus get another set of n quadric varieties *w.r.t.* each of which the simplexes $A_1A_2\ldots A_{n+1}$ and $A''_{11}A''_{12}\ldots A''_{1n+1}$ are reciprocal, the reciprocal of the vertex A_1 being the prime face of the latter simplex, which is opposite to the vertex A''_{1s}.

Hence, making A_1 correspond to each of the vertices $A''_{11}, A''_{12},\ldots A''_{1n+1}$ in turn, *we get in all $n(n+1)$ quadric varieties w.r.t. each of which the two simplexes $A_1A_2\ldots A_{n+1}$ and $A''_{11}A''_{12}\ldots A''_{1n+1}$ are reciprocal in some order.* And, in fact, it may be shown that *these are the only quadric varieties which reciprocate the two simplexes into one another, in some order.*

§ 13. Let us now denote the simplex $A''_{21}A''_{22}\ldots A''_{2n+1}$ of § 11 by S''_{11} when $x'_1=k_1$ and $x'_2=x'_3=\ldots=x'_{n+1}=1$, and by S''_{12} when $x'_1=k_2$ and $x'_2=x'_3=\ldots=x'_{n+1}=1$, where k_1 and k_2 are the two roots of equation (A), § 4. Consider first the simplex S''_{11}. It is obviously mutually self-polar *w.r.t.* the simplex of reference $A_1A_2\ldots A_{n+1}$. Further, if we transform the simplex of reference to the simplex $A''_{11}A''_{12}\ldots A''_{1n+1}$ of § 6, by means of the transformation

$$(\xi_1, \xi_2, \ldots \xi_{n+1}) = \begin{bmatrix} 1 & 1 & 1 & \ldots & \ldots & \ldots & 1 \\ 1 & \rho_1 & \rho_1^2 & \ldots & \ldots & \ldots & \rho_1^n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \rho_1^{n-1} & (\rho_1^{n-1})^2 & \ldots & \ldots & \ldots & (\rho_1^{n-1})^n \\ 1 & \rho_1^n & (\rho_1^n)^2 & \ldots & \ldots & \ldots & (\rho_1^n)^n \end{bmatrix} (x_1, x_2, \ldots x_{n+1})$$

where $\rho_1 = \dfrac{1}{\rho}$, then the co-ordinates of the vertices of S''_{11} are given by the rows of the determinant of §4, in which k is replaced by $\dfrac{1}{k_1}$.

It follows that the simplex S''_{11} is mutually self-polar, also *w.r.t.*, the simplex $A''_{11}A''_{12}\ldots A''_{1n+1}$. Similarly, the simplex S''_{12}, as defined in the beginning of the present article, is also mutually self-polar *w.r.t.* each of the two simplexes $A_1A_2\ldots A_{n+1}$ and $A''_{11}\,A''_{12}\ldots A''_{1n+1}$.

§ 14. The two simplexes S''_{11} and S''_{12} of the previous article were obtained from the simplex $A''_{21}\,A''_{22}\ldots A''_{2n+1}$ by putting the first co-ordinate of A''_{21} equal respectively to k_1 and k_2, and every other co-ordinate equal to unity. Similarly, by putting equal to unity, every co-ordinate of A''_{21} except one, which is put equal to k_1 and k_2 respectively, we get in all $n+1$ pairs of simplexes like the pair S''_{11} and S''_{12}.

By suitable transformations, the co-ordinates of the vertices of the two simplexes of each of the $n+1$ pairs, in relation to the two simplexes $A_1A_2\ldots A_{n+1}$ and $A''_{11}A''_{12}\ldots A''_{1n+1}$ can be put down in the same form as those of the vertices of the pair S''_{11} and S''_{12}. For example, the pair of simplexes S''_{r1} and S''_{r2} are obtained from the simplex $A''_{21}A''_{22}\ldots A''_{2n+1}$ by equating to unity every except the rth co-ordinate of A''_{21}, which is put respectively equal to k_1 and k_2. If then, we apply a transformation, in which the x_r, $x_{r+1}, \ldots x_{n+1}$, x_1, $x_2, \ldots x_{r-1}$ co-ordinates are changed respectively into the 1st, 2nd...,$(n+1)$th co-ordinates, the simplex of reference remaining the same, we see that the co-ordinates of the vertices of the simplexes $A''_{11}A''_{12}\ldots A''_{1n+1}$, S''_{r1} and S''_{r2} become the same as those of the vertices of the simplexes $A''_{11}A''_{12}\ldots A''_{1n+1}$, S''_{11} and S''_{12} considered in the previous article.

Thus each of the two simplexes of any one of the $n+1$ pairs, constructed above, is mutually self-polar *w.r.t.* each of the two simplexes $A_1A_2\ldots A_{n+1}$ and $A''_{11}A''_{12}\ldots A''_{1n+1}$. And it is clear that, in order to discuss the properties of the two simplexes of any one of the $n+1$ pairs, in regard to the two simplexes $A_1A_2 \ldots \ldots A_{n+1}$ and $A''_{11}A''_{12} \ldots\ldots A''_{1n+1}$ we need consider only the pair S''_{11} and S''_{12}.

§ 15. We now state and prove the following result:

The two simplexes of any one of the $n+1$ pairs, constructed in § 14, *are simultaneously in perspective with the simplex $A''_{11}A''_{12} \ldots A''_{1n+1}$ with one vertex of the simplex $A_1A_2 \ldots A_{n+1}$ as centre of perspective, and the cross-ratio of the four vertices of the four simplexes lying on each of the $n+1$ rays of perspective is constant, this constant being the same for all the $n+1$ pairs.*

For, consider the pair S''_{11}, S''_{12}. The two simplexes are simultaneously in perspective with $A''_{11}A''_{12} \ldots A''_{1n+1}$ with A_1 $(1, 0, 0, \ldots, 0)$ as the centre of perspective. The four points A_1, $(k_1, \rho^r, \rho^{2r}, \ldots, \rho^{nr})$ of S''_{11}, A''_{1r+1} and $(k_2, \rho^r, \rho^{2r}, \ldots, \rho^{nr})$ of S''_{12} lie on one of the $n+1$ rays of perspective through A_1, and their cross ratio is $(\infty,\ k_1-1,\ 0,\ k_2-1)=(k_2-1)/(k_1-1)$, which is a constant, being independent of r. That this constant is the same for all the $n+1$ pairs of simplexes, follows from the remark made towards the end of the previous article.

§ 16 The property of perspectivity, discussed in the previous article, is enough to determine the $n+1$ pairs of simplexes. In fact we have the following:

There are just $n+1$ pairs of simplexes which are mutually self-polar w.r.t. each of the two simplexes $A_1A_2 \ldots A_{n+1}$ and *$A''_{11}A''_{12} \ldots A''_{1n+1}$ and are such that the two simplexes of each pair are simultaneously in perspective with the simplex $A''_{11}A''_{12} \ldots A''_{1n+1}$ with one vertex of the simplex $A_1A_2 \ldots A_{n+1}$ as centre of perspective.*

For, the co-ordinates of the vertices of a simplex S which is in perspective with $A''_{11}A''_{12} \ldots A''_{1n+1}$ with A_1 as centre of perspective, must be given by the rows of the determinant

$$\begin{vmatrix} l_1 & 1 & 1 & \ldots & \ldots & \ldots & 1 & 1 \\ l_2 & \rho & \rho^2 & \ldots & \ldots & \ldots & \rho^{n-1} & \rho^n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ l_n & \rho^{n-1} & (\rho^{n-1})^2 & & \ldots & \ldots & (\rho^{n-1})^{n-1} & (\rho^{n-1})^n \\ l_{n+1} & \rho^n & (\rho^n)^2 & & \ldots & \ldots & (\rho^n)^{n-1} & (\rho^n)^n \end{vmatrix}$$

If, then, S is mutually self-polar *w.r.t.* $A_1A_2 \ldots A_{n+1}$ we must have $l_1/l_2=1$, etc., whence $l_1=l_2=l_3= \ldots =l_{n+1}=l$ say. If, further, S is mutually self-polar *w.r.t.* the simplex $A''_{11}A''_{12} \ldots A''_{1n+1}$, we see by applying transformations of § 18, that l must satisfy equation (A), § 4, *i.e.*, $l=k_1$ or k_2, giving the pair S''_{11} and S''_{12} as the two solutions for S. Considering all the vertices of the simplex of reference, in turn, as centres of perspective, the $n+1$ pairs of simplexes are thereby determined, which was to be proved.

§ 17. It may be observed that since the co-ordinates of the vertices of the simplex $A_1A_2 \ldots A_{n+1}$ when referred to the simplex $A''_{11}A''_{12} \ldots A''_{1n+1}$ may be put in the same form as those of the vertices of the latter referred to the former, we get $n+1$ pairs of simplexes, each of which is mutually self-polar *w.r.t.* each of the two simplexes $A_1A_2 \ldots A_{n+1}$ and $A''_{11}A''_{12} \ldots A''_{1n+1}$, the two simplexes of each pair being simultaneously in perspective with the simplex $A_1A_2 \ldots A_{n+1}$ with one vertex of the simplex $A''_{11}A''_{12} \ldots A''_{1n+1}$ as centre of perspective. The results of §§ 15, 16 apply equally to the present set of $n+1$ pairs of simplexes, the constant cross-ratio of § 15 being the same in the present case as in that article.

§ 18. Coming particularly to the space of four dimensions, let us consider the polar prime, or the polar hyperplane, as it may be called, of a point A'_1 *w.r.t.* a simplex $A_1A_2 \ldots A_5$. We take A'_1 as the unit point referred to this simplex. The polar hyperplane of A'_1 *w.r.t.* the simplex $A_1A_2 \ldots A_5$ has the equation $\sum_{r=1}^{5} x_r = 0$.

Now, let A'_1 and A'_2 be two vertices of a simplex which is self-polar *w.r.t.* the simplex $A_1A_2 \ldots A_5$. Then the polar hyperplanes of A'_1 and A'_2 must pass through the points A'_2 and A'_1 respectively. Hence the co-ordinates of the point A'_2 must satisfy the two equations $\Sigma\, x_r = 0$ and $\Sigma\, 1/x_r = 0$. It follows that the remaining four vertices of a simplex which has A'_1 as one vertex and is self-polar *w.r.t.* the simplex $A_1A_2 \ldots A_5$ lie on the surface given by the above two equations, which may also be written as $\Sigma\, x_r = 0$ and $(x_2x_3x_4 + x_3x_1x_4 + x_1x_2x_4 + x_1x_2x_3)(x_1 + x_2 + x_3 + x_4) = x_1x_2x_3x_4$. Let P, Q, R, S and T be respectively the points $(1, 0, 0, 0, -1)$, $(0, 1, 0, 0, -1)$, $(0, 0, 1, 0, -1)$, $(0, 0, 0, 1, -1)$ and $(1, 1, 1, 1, -4)$. They lie in the hyperplane $\Sigma\, x_r = 0$. Hence, in this hyperplane, referred to the tetrahedron PQRS, with T as the unit point, the surface in question is given by the single equation

$$(x_2x_3x_4 + x_3x_1x_4 + x_1x_2x_4 + x_1x_2x_3)(x_1 + x_2 + x_3 + x_4) = x_1x_2x_3x_4.$$

This is a quartic surface having ten double points which arrange themselves in a three-dimensional 10_3 *configuration of Desargue.*

The ten double points are $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(1, 0, 0, -1)$, $(0, 1, -1, 0)$, $(0, 1, 0, -1)$, $(1, 0, -1, 0)$, $(0, 0, 1, -1)$ and $(1, -1, 0, 0)$. Since each of the 10 lines of this 10_3 configuration passes through three double points of the surface, we see that *the ten lines of the configuration lie on the quartic surface.* We remark that the points, edges and planes of this configuration arise respectively as the sections by the hyperplane $\Sigma\, x_r = 0$ of the edges, plane and hyperplane faces of the simplex $A_1 A_2 \ldots A_5$. Since the co-ordinates of the ten points of this configuration can always be put down as above, it follows that with every three-dimensional 10_3 configuration of Desargue is associated a quartic surface as the one considered.

The reciprocal results may be similarly considered.

§ 19. As a particular case of §6, when ρ is an imaginary fifth root of unity, the simplex $A''_{11}A''_{12} \ldots A''_{15}$, the co-ordinates of whose vertices are given by the rows of the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \rho & \rho^2 & \rho^3 & \rho^4 \\ 1 & \rho^2 & \rho^4 & \rho & \rho^3 \\ 1 & \rho^3 & \rho & \rho^4 & \rho^2 \\ 1 & \rho^4 & \rho^3 & \rho^2 & \rho \end{vmatrix}$$

is mutually self-polar *w.r.t.* the simplex of reference $A_1 A_2 \ldots A_5$. It may be shown that the joins of the corresponding vertices of these two simplexes, namely, the lines $A_1 A''_{11}$, $A_2 A''_{12}, \ldots, A_5 A''_{15}$ form a set of five associated lines (Baker, p. 114). This is, in fact, contained in what we now state:

Pairs of points A_1, A''_{11} ; A_2, A''_{12} ; … ; A_5, A''_{15} *can be uniquely chosen respectively on the five lines a, b, c, d and e of an associated set such that the co-ordinates of the points* $A''_{11}\ A''_{12} \ldots A''_{15}$ *referred to the simplex* $A_1A_2 \ldots A_5$ *are given by the rows of the above determinant.*

Let the points of the line a from which transversals can be drawn respectively to the pairs of lines c, d ; d, b ; b, c be denoted by α. α_1 and $\bar{\alpha}$. Then it can be shown that $(\alpha_1, \bar{\alpha}, \alpha, A_1) = 1+\rho$, and $(\alpha_1, \bar{\alpha}, \alpha, A''_{11}) = 1+\rho^4$. By means of these and other similar cross-ratios, the pairs of points A_1, A''_{11}, etc., can be located geometrically.

§ 20. In the previous article we have shown how a set of five associated lines can be thought of as the joins of the corresponding vertices of a particular type of a pair of mutually self-polar simplexes. Reciprocally, we may show that *a set of five associated planes in a space of four dimensions* (Baker, p. 118) *arise as the intersections of the corresponding hyperplane faces of the same type of a pair of mutually self-polar simplexes.*

Gratitude is due to Prof. C. V. Hanumanta Rao, under whose guidance this paper was written.

PUNJAB COLLEGE OF ENGINEERING AND
TECHNOLOGY, LAHORE.

References

Baker, *Principles of Geometry*, Vol. IV.
Hameed, A., 1941, *Bull. Cal. Math. Soc.*, 33, 167-186.

SPONTANEOUS ENERGY FLUCTUATION AND QUANTUM STATISTICS

By

B. N. Biswas

(Communicated by the Secretary—Received January 6, 1943)

The purpose of the present paper is to derive expressions for—

(*a*) the energy fluctuation at constant volume of a perfect monatomic gas under conditions of negligible degeneracy,

(*b*) the fluctuation of the number per "S" type of cell in the cases of Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac distributions,

(*c*) the energy fluctuation of quanta, and

(*d*) the energy fluctuation of a monatomic solid, in the light of modified Gibb's statistics as developed in two previous papers (1931-32, 1938).

In these papers, it was shown that when statistical equilibrium exists amongst the members of the system, the density of distribution, *i.e.*, the density of a standard element of volume Δ_T, satisfying the differential equation

$$\Sigma\left(\frac{\partial\rho}{\partial q_1}\frac{\partial u}{\partial p_1}-\frac{\partial\rho}{\partial p_1}\frac{\partial u}{\partial q_1}\right)=0$$

has solutions $\rho=(N/A)\,e^{-\beta u}$ for photons, and $\rho=(N/A)\,e^{\beta(\psi-u)}$ for material particles, where $A=Nh^3$ and ψ, β are constants.

If gas molecules are represented in six-dimensional space, divided into cells of volume h^3, then on following the lines of the previous paper (1931-32), we have

$$C_{rs}=A_s\{1-e^{\beta(\psi-u_s)}\}e^{\gamma\beta(\psi-u_s)},\qquad r=0,1,\ldots\ldots$$

where

$$A_s=\frac{2\pi v\,(2m)^{\frac{3}{2}}u_s^{\frac{1}{2}}du_s}{h^3}$$

denotes the total number of cells in the phase space included between the energy layers u_s and u_s+du_s, and C_{rs} the number of cells in that domain of the phase-space including r-representative points The total number of R-points, N_s in the same domain, and the energy E_s, are given by

$$N_s=\Sigma rC_{rs}=\frac{A_s}{e^{-\beta(\psi-u_s)}-1},\quad E_s=\frac{A_su_s}{e^{-\beta(\psi-u_s)}-1}.$$

In order to evaluate β, we can put $W_s=A_s!/\pi C_r!$ as explained in the paper (1938), and with the help of Boltzmann's equation, it can be shown that $\beta=1/KT$.

(*a*) *Energy fluctuation at constant volume of a perfect monatomic gas under non-degeneracy.*

$$S = \text{entropy of the gas} = \Sigma S_s$$

where
$$S_s = K \log W_s = K \log \frac{A_s!}{\pi C_r!} .$$

$$\therefore\ S = -K \sum_s A_s \log \{1 - e^{(\psi - u_s)/\theta}\} + \frac{K}{\theta} \sum_s \frac{A_s u_s}{e^{-(\psi - u_s)/\theta} - 1}$$
$$- \frac{K\psi}{\theta} \sum_s \frac{A_s}{e^{-(\psi - u_s)/\theta} - 1}, \quad \text{where } \theta = KT.$$

$$\delta S = K \frac{\partial}{\partial \theta} \Sigma \{ A_s e^{(\ - u_s)/\theta} \} \delta\theta + \frac{K}{\theta} \frac{\partial}{\partial \theta} \Sigma \{ A_s u_s e^{(\psi - u_s)/\theta} \} \delta\theta$$
$$+ \sum_s \{ A_s u_s e^{(\psi - u_s)/\theta} \} \times \frac{\partial}{\partial \theta}\left(\frac{K}{\theta}\right) \delta\theta - \frac{K\psi}{\theta} \frac{\partial}{\partial \theta} \Sigma \{ A_s e^{(\psi - u_s)/\theta} \} \delta\theta$$
$$+ \Sigma \{ A_s e^{(\psi - u_s)/\theta} \} \frac{\partial}{\partial \theta} (-K\psi/\theta)\, \delta\theta = \frac{K}{\theta}\, \delta\overline{E}, \qquad \ldots \quad (1)$$

where $\overline{E}$ represents the total mean energy of the system.

$$\delta N = 0 = \delta\, \Sigma A_s e^{(\psi - u_s)/\theta}.$$

The first and the fourth terms of δS in (1) are zero. The second, third and the fifth terms are respectively

$$\frac{K}{\theta^3} \delta\theta (N\overline{U^2} - N\,\overline{U}^2), \quad -N\overline{U}.\ \frac{K}{\theta} \delta\theta \quad \text{and} \quad \frac{KN\overline{U}\delta\theta}{\theta^2},$$

where
$$\overline{U^2} = \frac{\Sigma A_s u_s^2 e^{(\psi - u_s)/\theta}}{\Sigma n_s}, \quad \overline{U} = \frac{\Sigma A_s u_s e^{(\psi - u_s)/\theta}}{\Sigma n_s}.$$

Hence
$$\frac{K}{\theta} \overline{\delta E} = \frac{K}{\theta^3} \delta\theta \left\{ N\overline{U^2} - N\,\overline{U}^2 \right\}.$$

Putting $\overline{E} = N\overline{U}$,

$$\frac{N\,\overline{U^2} - N\,\overline{U}^2}{N^2\,\overline{U}^2} = \frac{\theta^2}{N\,\overline{U}^2} \cdot \frac{d\overline{U}}{d\theta} = \frac{\overline{E^2} - \overline{E}^2}{\overline{E}^2} = (\overline{\delta^2 E})_v \ \text{(classical)} \qquad \ldots \quad (2)$$

and for ideal gas or colloidal particles

$$(\overline{\delta^2 E})_v = \frac{2}{3N} = \frac{\overline{U^2} - \overline{U}^2}{N\,\overline{U}^2}. \qquad \ldots \quad (3)$$

Calculation of the second order term in the expression for $(\overline{\delta^2 E})_v$

From the above consideration it is clear that

$$\frac{N\overline{U^2} - N\,\overline{U}^2}{N^2\,\overline{U}^2} = \frac{\overline{E^2} - \overline{E}^2}{\overline{E}^2}$$

up to the first order. In order to evaluate the second order term in the expression for the energy fluctuation at constant volume which is characteristic of two statistics, Bose-Einstein and Fermi-Dirac, it is assumed that the two are equal even up to the second order. The method of procedure is as follows:

$$(\overline{\delta^2 E})_v = \frac{\overline{E^2} - \overline{E}^2}{\overline{E}^2} = \frac{\overline{U^2} - \overline{U}^2}{N\,\overline{U}^2}.$$

$\overline{U^2}$ and $\overline{U}^2$ are to be evaluated according to the two statistics directly from the rule of averaging.

$$\overline{U^2} = \frac{\sum_s A_s u_s^2 e^{(\psi-u_s)/\theta}\{1 \pm e^{(\psi-u_s)/\theta}\}}{\sum_s n_s},$$

$$N\,\overline{U^2} = \frac{2\pi V(2m)^{\frac{3}{2}}}{h^3}[e^{\psi/\theta}\,\Gamma(\tfrac{7}{2})\theta^{\frac{7}{2}} \pm e^{2\psi/\theta}\Gamma(\tfrac{7}{2})\theta^{\frac{7}{2}}/2^{\frac{7}{2}}]. \qquad \ldots \quad (4)$$

The + sign corresponds to Bose-Einstein and the − sign to Fermi-Dirac statistics. Also

$$e^{\psi/\theta} = \frac{Nh^3}{V(2\pi m\theta)^{\frac{3}{2}}}(1 \mp y)$$

where $$y = Nh^3/V(4\pi m\theta)^{\frac{3}{2}}.$$

From (4), we have

$$N\,\overline{U^2} = \tfrac{15}{4}N\theta^2(1 \mp \tfrac{3}{4}y),\ N\,\overline{U} = \tfrac{3}{2}\,N\theta(1 \mp \tfrac{1}{2}y). \qquad \ldots \quad (5)$$

From (3) and (5), we have (Ghose, 1935)

$$\frac{N\,\overline{U^2} - N\,\overline{U}^2}{N^2\,\overline{U}^2} = \frac{2}{3N}\,(1 \pm \tfrac{5}{8}y). \qquad \ldots \quad (6)$$

Fürth's value for the above is $\frac{2}{3N}\,(1 \pm \frac{5}{4}y)$. This can be derived from (2) by putting the value of $\overline{U}$ as given by the second equation of (5). This clearly shows that some second order terms have been left out of calculation. Hence, according to our considerations, Fürth's result is incorrect.

(*b*) *Fluctuation within the cell.*

Mean square deviation or fluctuation is given by

$$\overline{\delta^2} N_s = \frac{\overline{N_s^2} - \overline{N}_s^2}{\overline{N}_s^2} \qquad \ldots \quad (7)$$

Bose-Einstein.

Mean square number per cell

$$= \overline{N_s^2} = \frac{e^{(\psi-u_s)/\theta}[1 + e^{(\psi-u_s)/\theta}]}{[1 - e^{(\psi-u_s)/\theta}]^2}. \qquad \ldots \quad (8)$$

Mean number of molecules per cell

$$=\overline{N}_s=\frac{1}{e^{-(\psi-u_s)/\theta}-1}. \qquad \dots \quad (9)$$

From (7), (8), (9) we have

$$\overline{\delta^2 N_s}=e^{-(\psi-u_s)/\theta}=\frac{1}{\overline{N}_s}+1. \qquad \dots \quad (10)$$

Fermi-Dirac.

$$C_0=\frac{A_s}{1+e^{(\psi-u_s)/\theta}},\ C_1=\frac{A_s e^{(\psi-u_s)/\theta}}{1+e^{(\psi-u_s)/\theta}}$$

from eq. (32) of a previous paper (1931-32).

$$\overline{N}_s=\frac{e^{(\psi-u_s)/\theta}}{1+e^{(\psi-u_s)/\theta}},\ \overline{N_s^2}=\frac{e^{(\psi-u_s)/\theta}}{1+e^{(\psi-u_s)/\theta}}.$$

We have, as before,

$$\overline{\delta^2 N_s}=\frac{1}{\overline{N}_s}-1. \qquad \dots \quad (11)$$

Hence from (10) and (11),

$$\overline{\delta^2 N_s}=\frac{1}{\overline{N}_s}\pm 1, \qquad \dots \quad (12)$$

the + sign for Bose and the − sign for Fermi statistics. Equation (12) is the same as obtained by Schrödinger (1926), Fürth (1928), Tolman (1938) and Ghose (1935).

Classical.

By carrying out the process as indicated above and neglecting unity compared with $e^{-(\psi-u_s)/\theta}$, we have

$$\overline{\delta^2 N_s}=\frac{1}{\overline{N^2}}, \qquad \dots \quad (13)$$

which is the same as given by Tolman (1938).

Mean square deviation is defined to be equal to

$$\overline{\Delta_s^2}=\frac{\overline{N_s^2}-\overline{N}_s^2}{A_s\overline{N}_s^2}=\frac{1}{N_s}\pm\frac{1}{A_s} \qquad \dots \quad (14)$$

on putting $A_s\overline{N}_s=N_s$. On putting $\psi=0$, equation (14) assumes the form after omission of the negative sign

$$\overline{\Delta_s^2}=\frac{1}{N_s}+\frac{1}{A_s} \qquad \dots \quad (15)$$

which is the same as that derived by Einstein giving number fluctuation of quanta.

(c) *Energy fluctuation of ν_s – radiation.*

In this case, energy fluctuation is defined to be equal to the dispersion of energy associated with the total S-type cells.

Dispersion of energy for ν_s – radiation

$$\overline{\delta^2 E_{\nu_s}} = A_s(\overline{U_s^2} - \bar{U}_s^2) \qquad \ldots \quad (16)$$

where A_s is the total number of cells in the energy layers $h\nu_s$ and $h(\nu_s + d\nu_s)$ in the corresponding phase-space of six dimensions representing the photons as explained in a previous paper (1938), $\overline{U_s^2}$ the mean square of energy per cell, $\bar{U}_s$ the mean energy per cell.

$$C_r = A_s(1 - e^{-h\nu_s/kT})e^{-rh\nu_s/kT}, \quad r = 0, 1, \ldots$$

$$\overline{U_s^2} = \frac{h^2\nu_s^2 e^{-h\nu_s/kT}[1 + e^{-h\nu_s/kT}]}{[1 - e^{-h\nu_s/kT}]^2}, \quad \bar{U}_s = \frac{h\nu_s e^{-h\nu_s/kT}}{1 - e^{-h\nu_s/kT}}. \qquad \ldots \quad (17)$$

Hence the fluctuation per unit volume as defined by (16)

$$= h\nu_s \rho_{(\nu_s)} d\nu_s + \frac{c^3\{\rho_{(\nu_s)}\}^2 d\nu_s}{8\pi\nu_s^2}, \qquad \ldots \quad (18)$$

where
$$\rho_{\nu_s} d\nu_s = \frac{8\pi\nu_s^2 h\nu_s d\nu_s}{c^3(e^{h\nu_s/kT} - 1)}.$$

Equation (18) is the same as obtained by Fowler (1929).

(d) *Energy fluctuation of a monatomic solid.*

In a recent paper (Biswas, 1943), photon statistics has been successfully applied to a monatomic solid to derive expressions for its C_v, entropy and free energy.

Hence
$$\overline{\delta^2 E_{\nu_s}} = A_s(\overline{U_s^2} - \bar{U}_s^2),$$

and on putting
$$A_s = 4\pi V\left\{\frac{1}{C_L^3} + \frac{2}{C_T^3}\right\}\nu_s^2 d\nu_s$$

in the case of a monatomic solid, we obtain the fluctuation of energy per unit volume

$$= 4\left\{\frac{1}{C_L^3} + \frac{2}{C_T^3}\right\} \cdot \frac{h^2\nu_s^4 e^{h\nu_s/kT} d\nu_s}{[e^{h\nu_s/kT} - 1]^2}. \qquad \ldots \quad (19)$$

RAJARAM COLLEGE,
KOLHAPUR.

References

Biswas, 1938, *J. Ind Math. Soc.* (New Series), **3**, 82.
Biswas, 1943, *Ind. Jour. Physics*, **17**.
Biswas and Sen, 1931-32, *J. Ind. Math Soc*, **19**, 130
Fowler, 1929, *Statistical Mechanics*, 514.
Fürth, 1928, *Zeit f. Phys.*, **48**, 325.
Ghosh, 1935, *Phil. Mag.*, **19**, 694.
Schrödinger, 1926, *Phys. Zeit.*, **27**, 95.
Tolman, 1938, The Principles of Statistical Mechanics, 630.

STRESS SYSTEMS IN ROTATING AEOLOTROPIC DISCS

By

S. Ghosh

(Received March 15, 1948)

This paper is a continuation of a previous one by the present author (1942), in which the conditions for the existence of a state of plane stress in an aeolotropic material with three perpendicular planes of symmetry at each point have been discussed. It has been found that if a plate of such a material, with its plane faces parallel to one of the planes of symmetry, be subjected to arbitrarily prescribed tractions on its rim, a state of plane stress cannot, in general, exist in the plate. Some simple examples of plane stress that can occur in the plate and each of which requires the tractions on the rim to be distributed in some restricted manner, have been considered in the paper mentioned above. The present paper deals with a plane stress problem in which body forces are present, *viz.*, that of the distribution of stress in a rotating aeolotropic disc with circular or elliptic boundary. The only complication that arises from the presence of the body forces is that the partial differential equation of the fourth order satisfied by the stress function is no longer homogeneous. In spite of this complication, the absence of tractions on the rim of the plate leads to an expression of the stress function which is a polynomial of the fourth degree in x, y and quadratic in z, where $z=0$ is the middle plane of the plate. This expression for the stress function is at once seen to satisfy the conditions (obtained in the previous paper) that are to be imposed on this function for the existence of a state of plane stress in the plate.

Fundamental Equations

Let us take the middle plane of the plate as the plane $z=0$, the axis of rotation as the axis of z and the directions of the lines of symmetry in the plane $z=0$ as the directions of the axes of x, y. Assuming that $X_z = Y_z = Z_z = 0$ throughout the plate, the equations of motion become

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = -\rho\omega^2 x, \quad \frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} = -\rho\omega^2 y,$$

where ω is the angular velocity of rotation. The stress components can therefore be expressed in terms of a stress function χ, by the formulae

$$X_x = \frac{\partial^2 \chi}{\partial y^2} - \tfrac{1}{2}\rho\omega^2 r^2, \quad Y_y = \frac{\partial^2 \chi}{\partial x^2} - \tfrac{1}{2}\rho\omega^2 r^2, \quad X_y = -\frac{\partial^2 \chi}{\partial x \partial y}, \qquad \text{..} \quad (1)$$

where $r^2 = x^2 + y^2$.

The stress-strain relations are (Ghosh, 1942, p. 163).

$$\left.\begin{aligned} e_{xx}&=s_{11}X_x+s_{12}Y_y,\ e_{yy}=s_{12}X_x+s_{22}Y_y,\ e_{xy}=s_{66}X_y, \\ e_{yz}&=e_{zx}=0,\ e_{zz}=s_{13}X_x+s_{23}Y_y. \end{aligned}\right\} \qquad \ldots \quad (2)$$

To satisfy these conditions, we assume

$$\left.\begin{aligned} u&=u_0-\tfrac{1}{2}z^2\frac{\partial w_1}{\partial x},\ v=v_0-\tfrac{1}{2}z^2\frac{\partial w_1}{\partial y}, \\ w&=w_1z+w_3z^3,\ \chi=\chi_0+\chi_2z^2, \end{aligned}\right\} \qquad \ldots \quad (3)$$

where, u_0, v_0, w_1, χ_0, χ_2 are independent of z and w_3 is a constant.

Substituting in (2) and eliminating u_0, v_0 from the three equations satisfied by them we see that χ_0 satisfies the equation

$$s_{22}\frac{\partial^4\chi_0}{\partial x^4}+(2s_{12}+s_{66})\frac{\partial^4\chi_0}{\partial x^2\partial y^2}+s_{11}\frac{\partial^4\chi_0}{\partial y^4}=\rho\omega^2(s_{11}+2s_{12}+s_{22}), \qquad \ldots \quad (4)$$

and we get from the other equations,

$$\left.\begin{aligned} &-\tfrac{1}{2}\frac{\partial^2 w_1}{\partial x^2}=s_{11}\frac{\partial^2\chi_2}{\partial y^2}+s_{12}\frac{\partial^2\chi_2}{\partial x^2},\ -\tfrac{1}{2}\frac{\partial^2 w_1}{\partial y^2}=s_{12}\frac{\partial^2\chi_2}{\partial y^2}+s_{22}\frac{\partial^2\chi_2}{\partial x^2}, \\ &\frac{\partial^2 w_1}{\partial x\partial y}=s_{66}\frac{\partial^2\chi_2}{\partial x\partial y},\ w_1=s_{13}\frac{\partial^2\chi_0}{\partial y^2}+s_{23}\frac{\partial^2\chi_0}{\partial x^2}-\tfrac{1}{2}\rho\omega^2(s_{13}+s_{23})r^2, \\ \text{and}\qquad &3w_3=s_{13}\frac{\partial^2\chi_2}{\partial y^2}+s_{23}\frac{\partial^2\chi_2}{\partial x^2}. \end{aligned}\right\} \quad (5)$$

Elliptic Disc

In the case of an elliptic plate with its semi-axes in the directions of the lines of symmetry and of lengths a, b and with its centre on the axis of z, we take

$$\left.\begin{aligned} \chi_0&=a_0x^2+b_0y^2+Ax^4+2Bx^2y^2+Cy^4, \\ \chi_2&=a_2x^2+b_2y^2. \end{aligned}\right\} \qquad \ldots \quad (6)$$

Since χ_0 satisfies the equation (4), we have

$$s_{22}A+\tfrac{1}{3}(2s_{12}+s_{66})B+s_{11}C=\tfrac{1}{24}\rho\omega^2(s_{11}+2s_{12}+s_{22}). \qquad \ldots \quad (7)$$

Let ν be the normal to the rim of the plate, so that its direction cosines are $\frac{px}{a^2}$, $\frac{py}{b^2}$, 0, where $\frac{1}{p^2}=\frac{x^2}{a^4}+\frac{y^2}{b^4}$. Then

$$X_\nu=\frac{px}{a^2}X_x+\frac{py}{b^2}X_y,\quad Y_\nu=\frac{px}{a^2}X_y+\frac{py}{b^2}Y_y.$$

Calculating X_x, Y_y, X_y from (1) and (6) on the boundary $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$$X_\nu=\frac{px}{a^2}\{2b_0+(4B-\tfrac{1}{2}\rho\omega^2)a^2+2b_2z^2\}+pxy^2\left\{\frac{12C}{a^2}-\frac{12B}{b^2}+\tfrac{1}{2}\rho\omega^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\right\},$$

$$\mathrm{Y}_\nu = \frac{py}{b^2}\{2a_0+(4\mathrm{B}-\tfrac{1}{2}\rho\omega^2)b^2+2a_2z^2\}+px^2y\left\{\frac{12\mathrm{A}}{b^2}-\frac{12\mathrm{B}}{a^2}-\tfrac{1}{2}\rho\omega^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\right\}.$$

We now make the coefficient of xy^2 in X_ν and x^2y in Y_ν vanish, so that

$$\mathrm{A}a^2=\mathrm{B}b^2+\tfrac{1}{24}\rho\omega^2(a^2-b^2),\ \mathrm{C}b^2=\mathrm{B}a^2-\tfrac{1}{24}\rho\omega^2(a^2-b^2). \qquad \dots \quad (8)$$

The equations (7) and (8) give

$$\mathrm{B} = \tfrac{1}{8}\rho\omega^2\frac{s_{11}a^4+2s_{12}a^2b^2+s_{22}b^4}{3(s_{11}a^4+s_{22}b^4)+(2s_{12}+s_{66})a^2b^2}. \qquad \dots \quad (9)$$

Calculating w_1 from (6) and the last equation of (5) and substituting in the other equations of (5), we get two equations for the determination of a_2, b_2. They give

$$\left.\begin{aligned}a^2b^2(s_{11}s_{22}-s_{12}{}^2)a_2 &= \tfrac{1}{4}\rho\omega^2(s_{13}a^2+s_{23}b^2)(s_{11}a^2-s_{12}b^2)\\ &\quad -2\{(3s_{13}a^2+s_{23}b^2)s_{11}a^2-(s_{13}a^2+3s_{23}b^2)s_{12}b^2\}\mathrm{B},\\ a^2b^2(s_{11}s_{22}-s_{12}{}^2)b_2 &= -\tfrac{1}{4}\rho\omega^2(s_{13}a^2+s_{23}b^2)(s_{12}a^2-s_{22}b^2)\\ &\quad +2\{(3s_{13}a^2+s_{23}b^2)s_{12}a^2-(s_{13}a^2+3s_{23}b^2)s_{22}b^2\}\mathrm{B}.\end{aligned}\right\} \dots \quad (10)$$

On the rim of the plate,

$$\int_{-h}^{h}\mathrm{X}_\nu dz=0, \quad \int_{-h}^{h}\mathrm{Y}_\nu dz=0,$$

so that

$$a_0+(2\mathrm{B}-\tfrac{1}{4}\rho\omega^2)b^2+\tfrac{1}{3}h^2a_2=0, \quad b_0+(2\mathrm{B}-\tfrac{1}{4}\rho\omega^2)a^2+\tfrac{1}{3}h^2b_2=0, \quad \dots \quad (11)$$

which determine a_0, b_0.

The average values of the stresses on the rim are

$$\overline{\mathrm{X}}_x=\frac{8\mathrm{B}a^2y^2}{b^2}, \quad \overline{\mathrm{Y}}_y=\frac{8\mathrm{B}b^2x^2}{a^2}, \quad \overline{\mathrm{X}}_y= 8\mathrm{B}xy. \qquad \dots \quad (12)$$

Let s be the tangent to the rim of the plate. Calculating $\widehat{ss}$ from the average stresses (12) on the rim, we have

$$\widehat{ss}=\frac{8\mathrm{B}a^2b^2}{p^2}, \qquad \dots \quad (13)$$

where p is the perpendicular from the centre of the plate on the tangent to the rim at the point. $\widehat{ss}$ is greatest when p is least, *i.e.*, at the extremity of the minor axis and the greatest value is

$$8\mathrm{B}a^2=\frac{\rho\omega^2a^2}{3+\alpha}, \qquad \dots \quad (14)$$

where

$$\alpha=\frac{(s_{66}-4s_{12})a^2b^2}{s_{11}a^4+2s_{12}a^2b^2+s_{22}b^4}. \qquad \dots \quad (15)$$

If the major axis of the elliptic boundary coincides with the y-axis, the greatest value of $\widehat{ss}$ (which occurs at the extremity of the minor axis) is

$$\frac{\rho\omega^2a^2}{3+\beta}, \qquad \dots \quad (16)$$

where
$$\beta = \frac{(s_{66}-4s_{12})a^2b^2}{s_{22}a^4+2s_{12}a^2b^2+s_{11}b^4}. \qquad \ldots \quad (17)$$

The constants s_{11}, s_{22}, s_{66} are, in general, positive and s_{12} negative. If $s_{11} > s_{22}$, we have

$$\frac{1}{\alpha}-\frac{1}{\beta} = \frac{(s_{11}-s_{12})(a^4-b^4)}{(s_{66}-4s_{12})a^2b^2} > 0, \qquad \text{or } \alpha < \beta.$$

Therefore the greatest value of $\widehat{ss}$ given by (16) is less than the greatest value (14).

For applications, let us consider sheets which are cut out from Oak parallel to the grains of the wood, the direction of the fibres coinciding with that of the y-axis. The elastic constants for Oak, as given by Hörig (1935), are $s_{11}=10.15$, $s_{22}=1.72$, $s_{12}=-0.87$, $s_{66}=12.8$. With these values,

$$\alpha=\frac{16.28\,a^2b^2}{10.15\,a^4-1.74\,a^2b^2+1.72\,b^4}; \quad \beta=\frac{16.28\,a^2b^2}{1.72\,a^4-1.74\,a^2b^2+10.15\,b^4}.$$

In the case of an elliptic plate of an isotropic material of the same density with $\sigma=\frac{1}{4}$, we have for the greatest value of $\widehat{ss}$,

$$\frac{\rho\omega^2a^2}{3+\gamma},$$

where
$$\gamma = \frac{7\,a^2b^2}{2\,a^4+2b^4-a^2b^2}.$$

Now $\beta > \gamma$ when $20.52\,a^4-4.10\,a^2b^2-38.49\,b^4 > 0$.

Taking $f(x)=20.52-4.10\,x-38.49\,x^2$ where $x=\dfrac{b^2}{a^2}$, we have

$$f'(x)=-4.10-76.98\,x,$$

which is negative when x is positive.

Therefore $f(x)$ is positive when $x=0$, diminishes continually as x increases and becomes negative when $x=1$. Further, $f(x)$ vanishes when $x=0.67$ or $b/a=0.8$.

Hence, so long as $b/a < 0.8$, β is $> \gamma$, so that the greatest of $\widehat{ss}$ on the rim of the plate, when the minor axis of the ellipse is perpendicular to the fibres, is less than the greatest value of $\widehat{ss}$ on the rim of the corresponding isotropic plate.

We also observe that for α to be $> \gamma$, we must have

$$-38.49\,a^4-4.10\,a^2b^2+20.52\,b^4 > 0,$$

which cannot be the case when $a \geqq b$. Thus $\alpha < \gamma$, and therefore the greatest value of $\widehat{ss}$ on the rim of the plate when the major axis is perpendicular to the fibres is always greater than the greatest value of $\widehat{ss}$ on the rim of the corresponding isotropic plate.

Circular Plate

Putting $a=b$, we get the case of a circular plate. We have

$$\left.\begin{aligned}\chi_0&=a_0x^2+b_0y^2+Br^4\\ \chi_2&=a_2x^2+b_2y^2\end{aligned}\right\}$$

where
$$B=\tfrac{1}{8}\rho\omega^2\;\frac{s_{11}+2s_{12}+s_{22}}{3(s_{11}+s_{22})+2s_{12}+s_{66}}.$$

The average stresses on the boundary $r=a$, are

$$\overline{X}_x=8By^2,\quad \overline{Y}_y=8Bx^2,\quad \overline{X}_y=-8Bxy.$$

The average circumferential stress on the rim is $8Ba^2$.

References

Ghosh, S., 1942, *Bull. Cal. Math Soc.*, **34**, 157-169
Hörig, H., 1935, *Ingen. Arch.*, **6**, 8.

SOME SELF-RECIPROCAL FUNCTIONS

By

H. C. Gupta

(Communicated by the Secretary—Received April 1, 1943)

1. A function is said to be self-reciprocal in the Hankel Transform of order ν, if it satisfies the integral equation

$$f(x)=\int_0^\infty \sqrt{xy}\ \mathrm{J}_\nu(xy)\, f(y)\, dy. \qquad (1.0)$$

Following Hardy and Titchmarsh we shall call such functions R_ν. A number of functions $\mathrm{P}(x)$, known as kernels, has been given from time to time to transform an R_μ function $f(x)$ into an R_ν function $g(x)$, and *vice-versa*, by means of the equation

$$g(x)=\int_0^\infty \mathrm{P}(xy)\, f(y)\, dy.$$

The object of this paper is to use the following known kernels (Mohan, 1941) in investigating some new R_ν functions:

(*i*) $x^{\frac{3}{2}-\nu}\mathrm{J}_{\nu-\frac{1}{2}}(\frac{1}{2}x)\ \mathrm{J}_{\nu-\frac{3}{2}}(\frac{1}{2}x)$, transforming $\mathrm{R}_{3\nu-3}$ into R_ν and *vice-versa*,

(*ii*) $x^{\frac{1}{2}-\nu}\left[\mathrm{J}_{\nu-\frac{1}{2}}(\frac{1}{2}x)\right]^2$, transforming $\mathrm{R}_{3\nu-1}$ into R_ν and *vice-versa*,

(*iii*) $\mathrm{J}_{\frac{1}{2}\nu-\frac{1}{4}}(\frac{1}{2}x)\ \mathrm{J}_{\frac{1}{4}-\frac{1}{2}\nu}(\frac{1}{2}x)$, transforming $\mathrm{R}_{1-\nu}$ into R_ν and *vice-versa*.

The three kernels have been applied to the $\mathrm{R}_{\rho+2m}$ function (Varma, 1938)

$$\rho!\ x^{2m+\rho+\frac{1}{2}}\ e^{-\frac{1}{2}x^2}\ \mathrm{L}_\rho^m(\tfrac{1}{2}x^2)$$

$$=2^{\frac{1}{2}m+\frac{1}{2}}\, x^{m+\rho-\frac{1}{2}}\, e^{-\frac{1}{4}x^2}\ \mathrm{W}_{\rho+\frac{1}{2}+\frac{1}{2}m,\ \frac{1}{2}m}(\tfrac{1}{2}x^2) \qquad \dots \quad (1.1)$$

where $\rho+2m>-1$, and ρ is an even integer or zero. It is interesting to note that the functions investigated in this paper are in the form of generalised hypergeometric function ${}_2\mathrm{F}_2$.

2. To apply the three kernels to the $\mathrm{R}_{\rho+2m}$ function (1.1), we first evaluate the general integral

$$\mathrm{I}=\int_0^\infty \mathrm{J}_\mu(\tfrac{1}{2}xy)\ \mathrm{J}_\nu(\tfrac{1}{2}xy)\ (\tfrac{1}{4}xy)^\lambda y^{m+\rho-\frac{1}{2}}\, e^{-\frac{1}{4}y^2}\ \mathrm{W}_{\rho+\frac{1}{2}(m+1),\ \frac{1}{2}m}(\tfrac{1}{2}y^2)\, dy.$$

Since $J_{\mu}(x)J_{\nu}(x)=\frac{(\frac{1}{2}x)^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)}\,{}_2F_3\left(\begin{matrix}\frac{\mu+\nu+1}{2},\ \frac{\mu+\nu+2}{2}\ ;-x^2\\ \mu+1,\ \nu+1,\ \mu+\nu+1\ ;\end{matrix}\right)$,

we get, on using the following integral due to Goldstein (1932),

$$\int_0^\infty x^{l-1}e^{-\frac{1}{2}x}\,W_{k,\,m}(x)dx=\frac{\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)},\quad R(l\pm m+\tfrac{1}{2})>0,$$

that

$$I=\int_0^\infty\left[\sum_{r=0}^{\infty}\frac{(-)^r\Gamma(\mu+\nu+2r+1)(\frac{1}{2}xy)^{\mu+\nu+\lambda+2r}}{r!\,\Gamma(\mu+r+1)\Gamma(\nu+r+1)\Gamma(\mu+\nu+r+1)}\right]\times$$

$$\times\,y^{m+\rho-\frac{1}{2}}e^{-\frac{1}{4}y^2}W_{\rho+\frac{m+1}{2},\,\frac{m}{2}}(\tfrac{1}{2}y^2)\,dy \qquad \ldots \quad (2.1)$$

$$=\sum_{r=0}^{\infty}\frac{(-)^r\Gamma(\mu+\nu+2r+1)\Gamma\{\frac{1}{2}(\mu+\nu+\rho+\lambda)+\frac{3}{4}+m+r\}\Gamma\{\frac{1}{2}(\mu+\nu+\lambda+\rho)+\frac{3}{4}+r\}}{r!\,\Gamma(\mu+r+1)\Gamma(\nu+r+1)\Gamma(\mu+\nu+r+1)\Gamma\{\frac{1}{2}(\mu+\nu+\lambda-\rho)+\frac{3}{4}+r\}}\times$$

$$\times\frac{1}{(\sqrt{2})^{\frac{3}{2}-m-\rho}}\left(\frac{x}{2\sqrt{2}}\right)^{\lambda+\mu+\nu+2r} \qquad \ldots \quad (2.2)$$

$$=\frac{\Gamma\{\frac{1}{2}(\mu+\nu+\rho+\lambda)+\frac{3}{4}+m\}\Gamma\{\frac{1}{2}(\mu+\nu+\lambda+\rho)+\frac{3}{4}\}}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma\{\frac{1}{2}(\mu+\nu+\lambda-\rho)+\frac{3}{4}\}(\sqrt{2})^{\frac{3}{2}-m-\rho}}\cdot\left(\frac{x}{2\sqrt{2}}\right)^{\lambda+\mu+\nu}\times$$

$$\times\,{}_4F_4\left(\begin{matrix}\frac{\mu+\nu+1}{2},\ \frac{\mu+\nu+2}{2},\ \frac{\mu+\nu+\lambda+\rho+\frac{3}{2}+2m}{2},\ \frac{\mu+\nu+\lambda+\rho+\frac{3}{2}}{2},\\ \mu+1,\ \nu+1,\ \mu+\nu+1,\ \frac{1}{2}(\mu+\nu+\lambda+\frac{3}{2}-\rho)\ ;\ -\frac{1}{2}x^2\end{matrix}\right) \qquad \ldots \quad (2.3)$$

provided that $R(\lambda+\mu+\nu+m+\rho)>|m|-\frac{3}{2}$.

The term by term integration involved in (2.2) is justified, for—

(*i*) $y^{-\mu-\nu}J_\mu(\frac{1}{2}xy)\,J_\nu(\frac{1}{2}xy)$ represents an integral function and is consequently uniformly convergent in any arbitrary interval (0, A), and

(*ii*) the integral

$$\int_0^\infty\left|\sum_{r=0}^{\infty}\frac{\Gamma(\mu+\nu+2r+1)(\frac{1}{2}xy)^{\mu+\nu+2r+\lambda}}{\Gamma(\mu+r+1)\Gamma(\nu+r+1)\Gamma(\mu+\nu+r+1).r!}\right|\times$$

$$\times\left|e^{-\frac{1}{4}y^2}W_{\rho+\frac{m+1}{2},\,\frac{m}{2}}(\tfrac{1}{2}y^2)\,y^{m+\rho-\frac{1}{2}}\right|dy$$

converges under the condition stated on account of the asymptotic behaviour

$$W_{k,m}(x)\sim O(x^k e^{-\frac{1}{2}x}).$$

3. Using the first kernel with the $R_{3\nu-3}$ function (1.1), we obtain the R_ν function

$$\int_0^\infty(xy)^{\frac{3}{2}-\nu}J_{\nu-\frac{1}{2}}(\tfrac{1}{2}xy)\,J_{\nu-\frac{3}{2}}(\tfrac{1}{2}xy)\,y^{3\nu-m-\frac{7}{2}}e^{-\frac{1}{4}y^2}\,W_{3\nu-\frac{3}{2}m-\frac{5}{2},\,\frac{1}{2}m}(\tfrac{1}{2}y^2)\,dy.$$

The value of the integral is easily obtained from the general integral (2.3) by putting respectively $\frac{3}{2}-\nu$, $\nu-\frac{1}{2}$, $\nu-\frac{3}{2}$ and $3\nu-3-2m$ for λ, μ, ν and ρ; we thus obtain the R_ν function

$$f(x) \equiv x^{\nu-\frac{1}{2}} {}_2F_2(\nu,\ 2\nu-m-1;\ \nu+\tfrac{1}{2},\ m+2-\nu;\ -\tfrac{1}{2}x^2), \qquad \dots (3.1)$$

where (*i*) $\frac{3}{2}(\nu-1)-m$ is zero or a positive integer, and (*ii*) $\frac{1}{3}<R(2\nu-1)>R(m)$; and (*iii*) the integral (1.0) converges. By the principle of analytic continuation the function is R_ν under the less restrictive conditions (*i*) and (*iii*), *viz.*, $R(\frac{2}{3}m+\frac{1}{2})<R(\nu)>-\frac{1}{2}$, as may be inferred from the asymptotic behaviours of $J_\nu(x)$ and ${}_2F_2$, *viz.*, $J_\nu(x) \sim x^{-\frac{1}{2}}\cos(x+C)$ and (Wright, 1940);

$${}_2F_2(a,\ b,\ c,\ d;\ -x) \sim Ax^{-a}+Bx^{-b},\ R(x) \geq 0,$$

where A, B, C are some constants.

If, however, we apply the kernel to the corresponding R_ν function (1.1), we then obtain the $R_{3\nu-3}$ function

$$x^{\nu-\frac{1}{2}} {}_2F_2(\nu,\ \nu+\tfrac{1}{2}-m;\ 2\nu-1,\ m+\tfrac{1}{2};\ -\tfrac{1}{2}x^2)$$

valid, by the principle of analytic continuation, when $\frac{1}{2}\nu-m=0$ or a positive integer, $R(\nu) \geq \frac{1}{2}$ and $R(\nu-2m) \geq -\frac{3}{2}$.

4. The second kernel, when applied to the R_ν function (1.1), yields the $R_{3\nu-1}$ function

$$\int_0^\infty (xy)^{\frac{1}{2}-\nu}\left[J_{\nu-\frac{1}{2}}(\tfrac{1}{2}xy)\right]^2 y^{\nu-m-\frac{1}{2}} e^{-\frac{1}{4}y^2} W_{\nu-\frac{3}{2}m+\frac{1}{2},\ \frac{1}{2}m}(\tfrac{1}{2}y^2)\,dy.$$

To evaluate this integral put $\frac{1}{2}-\nu$, $\nu-\frac{1}{2}$, $\nu-\frac{1}{2}$ and $\nu-2m$ for λ, μ, ν and ρ respectively in the general integral (2,3) ; we thus obtain the $R_{3\nu-1}$ function

$$x^{\nu-\frac{1}{2}} {}_2F_2(\nu,\ \tfrac{1}{2}+\nu-m;\ 2\nu,\ m+\tfrac{1}{2};\ -\tfrac{1}{2}x^2)$$

valid, by analytic continuation, when $\frac{1}{2}\nu-m$ is zero or a positive integer and $R(2m-\frac{3}{2})<R(\nu)>0$.

Applying the kernel alternatively to $R_{3\nu-1}$ function corresponding to (1.1) we get again the R_ν function (3.1).

5. For the application of the third kernel, we put 0, $\frac{1}{4}-\frac{1}{2}\nu$, $\frac{1}{2}\nu-\frac{1}{4}$ and $1-\nu-2m$ respectively for λ, μ, ν and ρ in (2.3), and obtain the R_ν function

$${}_2F_2(\tfrac{1}{2},\ \tfrac{5}{4}-\tfrac{1}{2}\nu-m;\ \tfrac{1}{2}\nu+\tfrac{3}{4},\ m+\tfrac{1}{4}+\tfrac{1}{2}\nu;\ -\tfrac{1}{2}x^2)$$

valid, by analytic continuation, when $2-\nu-2m$ is an odd positive integer and $R(\frac{5}{2}-2m)>R(\nu)>-\frac{3}{2}$.

Using the kernel alternatively as before, we get again the same function.

It is interesting to note that the four self-reciprocal functions of the type ${}_2F_2$ investigated in this paper reduce to ${}_1F_1$ in quite a number of special cases, some of which are also particular cases of the R_ν function (Mohan, 1939)

$$x^{\nu+2n+\frac{1}{2}}e^{-\frac{1}{2}x^2}\,{}_1F_1(-n,\ n+\nu+1,\ \tfrac{1}{2}x^2),$$
$$R(\nu+2n)\geq-\tfrac{3}{2},\ R(\nu+n)>-1.$$

My best thanks are due to Dr. R. S. Varma, D.Sc., for suggesting the problem and for his interest in the paper.

CHRIST CHURCH COLLEGE
CAWNPORE

References

Goldstein, S., 1932, *Proc. Lond. Math. Soc.* (2), **34**. 106.
Mohan, B., 1939, *Quart. Jour. Math.* (Oxford) (10), **10**, 256.
Do. 1941, *Ind.-Jour. Physics*, **15**, 337-42.
Varma, R. S., 1938, *Jour. Ind. Math. Soc.*, **7**.
Wright, E. M., 1940, *Proc. Lond. Math.-Soc.* (2), **42**, 389.

ON HARMONIC FUNCTION

By

H. Sircar

(Received April 5, 1943)

It has been possible to find, thanks to Villat (1916), in a circular area bounded by a chord, an analytic function $f(z)=u+iv$, whose real part u has a prescribed value on the bounding chord and the normal derivative $\frac{\partial u}{\partial n}$ of whose real part vanishes on the circular boundary. Indeed, a function, with limited arbitrariness is involved therein, and an application to the elusive hydrodynamical problem of transversal oscillation of water across a long canal of circular section would be attempted in a future issue of the journal.

As any segment of a circle containing an angle β and bounded by a chord with extremities $w=\pm c$ in the w-plane can be conformally transformed into a semi-circle of unit radius with a bounding diameter of which the extremities are $z=\pm 1$ in the z-plane, by the substitution

$$\frac{w-c}{w+c}=\left(\frac{z-1}{z+1}\right)^{2\left(1-\frac{\beta}{\pi}\right)} e^{-i(\pi-2\beta)},$$

we need confine ourselves to a semi-circular cross-section of unit radius in the z-plane.

2. Take a rectangle of arbitrary but finite dimensions, $a=w_1$, $b=\frac{w_3}{i}$ so that w_3 is pure imaginary.

H. Villat has shown that the function

$$E(w)=\frac{iw_1}{\pi^2}\int_0^{2\pi} f(\epsilon)\left[\zeta\left(\frac{w_1}{i\pi}\log w-\frac{w_1}{\pi}\epsilon\right)(w_1,2w_3)-\zeta_3\left(\frac{w_1}{i\pi}\log w-\frac{w_1}{\pi}\epsilon\right)(w_1,2w_3)\right]d\epsilon,$$

defined in the couronne of extreme radii $e^{-\frac{2\pi}{i}\frac{w_3}{w_1}}$ and I has a real part equal to $f(\theta)$ at $w=e^{i\theta}$ of the unit circle and the imaginary part zero over the concentric circle $|w|=q=e^{-\frac{\pi}{i}\frac{w_3}{w_1}}$, where ζ is a Weierstrassian function with the semi-periods w_1, $2w_3$, and $\log w$ has its principal value.

This function is regular from two sides of $|w|=q$ and the normal derivative of the real part on the circle (q) is consequently zero.

Let $$f(2\lambda-\epsilon)=f(\epsilon).$$

Then

$$\mathrm{E}(w)=\frac{iw_1}{\pi^2}\int_0^\pi f(\epsilon)\left[\zeta\left(\frac{w_1}{i\pi}\log w-\frac{w_1}{\pi}\epsilon\right)+\zeta\left(\frac{w_1}{i\pi}\log w+\frac{w_1}{\pi}\epsilon\right)\right.$$
$$\left.-\zeta_3\left(\frac{w_1}{i\pi}\log w-\frac{w_1}{\pi}\epsilon\right)-\zeta_3\left(\frac{w_1}{i\pi}\log w+\frac{w_1}{\pi}\epsilon\right)\right]d\epsilon.$$

For real positive w (between q and 1), this is evidently real. For negative w between -1 and $-q$, $\log w=\log|w|+i\pi$, and for this value of the logarithm,

$$\mathrm{E}(w)=\frac{iw_1}{\pi^2}\int_0^\pi f(\epsilon)\left[\zeta_1\left(\frac{w_1}{i\pi}\log\ w|-\frac{w_1}{\pi}\epsilon\right)+\zeta_1\left(\frac{w_1}{i\pi}\log|w|+\frac{w_1}{\pi}\epsilon\right)\right.$$
$$\left.-\zeta_2\left(\frac{w_1}{i\pi}\log|w|-\frac{w_1}{\pi}\epsilon\right)-\zeta_2\left(\frac{w_1}{i\pi}\log|w|+\frac{w_1}{\pi}\epsilon\right)\right]d\epsilon$$

which is clearly real.

Thus, $\mathrm{E}(w)$ has a real part equal to $f(\theta)$ at $w=e^{i\theta}$ of the unit circle and its imaginary part zero over the remainder of the semi-couronne on the upper half of the w-plane (and therewith the normal derivative of its real part is zero there).

8. Let us effect the following transformation

$$w=qe^{\frac{\pi}{w_1}u}.$$

To the semi-couronne will correspond a rectangular domain two of whose sides the axes of co-ordinates of the u-plane will carry in their respective directions, the dimensions of the rectangle being $\frac{w_3}{i}$ along the positive real axis and w_1 along the positive imaginary axis. The unit circle corresponds to the $\mathrm{R}(u)=\frac{w_3}{i}$; the circle $|w|=q$, to $\mathrm{R}(u)=0$; while the line $\arg w=0$, corresponds to $\vartheta(u)=0$ and the line $\arg w=0$, to the line $\vartheta((u)=w_1$.

Let $$w'_1=w_3/i \text{ and } \frac{w'_3}{i}=w_1.$$

Let ν_1 be real number such that $0<\nu_1<2w'_1=\frac{2w_3}{i}$. The substitution $u_1+w'_1+\frac{\nu_1}{2}=u$ would conformally displace the rectangle so as to occupy a position in the u_1-plane with vertices

$$\frac{-\nu_1}{2},\ -\frac{\nu_1}{2}+w'_3,\ \frac{-\nu_1}{2}-w'_1+w'_3,\ \frac{-\nu_1}{2}-w'_1.$$

Let S_1, S_2, S_∞, S_4 be the elementary symmetric functions of four real numbers x_1, x_2, x_3, x_4 such that the relations

$$\wp(\nu_1/w'_1 w'_3)=\frac{S_1^2}{16}-\frac{S_2}{6}\,,\text{ and }\wp'(\nu_1/w'_1,\,w'_3)=-\frac{S_3}{4}+\frac{S_1S_2}{8}-\frac{S_1^3}{32}$$

are satisfied, in which S_1 is chosen quite arbitrarily, and $\wp$ is the Weierstranian elliptic function.

The transformation

$$u_2=\frac{S_1}{4}+\frac{1}{2}\,\frac{\wp'(u_1/w'_1,\,w'_3)-\wp'(\nu_1/w'_1,\,w'_3)}{\wp(u_1/w'_1,\,w'_3)-\wp(\nu_1/w'_1 w'_3)},$$

which carries

$$\sqrt{(u_2-x_1)(u_2-x_2)(u_2-x_3)(u_2-x_4)}=\wp(u_1/w'_1,\,w'_3)-\wp(u'_1+\nu_1/w'_1 w'_3),$$

and of which the inverse is

$$u_1=\int_{\infty}^{u_2}\frac{du_2}{\sqrt{(u_2-x_1)(u_2-x_2)(u_2-x_3)(u_2-x_4)}},$$

would conformally transform the rectangular domain in the u_1-plane into the lower half of the u_2-plane, the vertices would correspond to the points $u_2=x_r$ $(r=1,\,2,\,3,\,4)$ and in particular the edge which corresponds to the unit circle would be transformed into the portion of the u_2-real axis between x_2 and x_1; for,

$$x_1=2\zeta(\nu_1/2)-\zeta(\nu_1)+S_1/4;\ x_2=2\zeta_3(\nu_1/2)-\zeta(\nu_1)+S_1/4;$$

$$x_3=2\zeta_2(\nu_1/2)-\zeta(\nu_1)+S_1/4;\ x_4=2\zeta_1(\nu_1/2)-\zeta(\nu_1)+S_1/4;$$

w'_1, w'_1 being the semi-periods, and consequently

$$x_1>x_2>x_3>x_4$$

as

$$\zeta(u)>\zeta_3(u)>\zeta_2(u)>\zeta_1(n),\qquad (0<u<w'_1).$$

N.B.—$\wp(u+\nu_1)-\wp(u)$ is a doubly periodic elliptic function having four poles, *viz.*, $-\nu_1$, $-\nu_1$, 0, 0, within the displaced period-parallelogram $(2w'_1,\,2w'_3)$ and consequently would have four zeros within the same domain; these zeros are evidently

$$-\nu_1/2,\ -\nu_1/2+w'_3,\ -\frac{\nu_1}{2}-w'_1+w'_3,\ -\nu_1/2-w'_1,$$

The formula $\zeta(u/w'_1,\,w'_3)=i\zeta(iu/w_1,\,w_3)$ may be utilised when a change of periods becomes necessary.

The transformation

$$u_3=-A\,\frac{u_2-x_2}{u_2-x_1}\ (A>0)=-\frac{u_2-x_2}{u_2-x_1}\ \text{(A is taken to be equal to 1)}$$

would transform the lower u_2-half-plane into the lower u_3-half-plane while $x_1\longrightarrow\infty$ and $x_2\longrightarrow0$ in the u_3-plane.

The substitution $u_3=u_4^2$ would transform the u_3-lower-half-plane into the lower right u_4 quarter-plane or upper-left-quarter-plane. We chose the former.

The substitution

$$u_4=-\lambda\,\frac{z+1}{z-1}\ (\lambda>0)=-\frac{z+1}{z-1}\ (\lambda\text{ is taken to be equal to 1})$$

would transform the u_4-lower right quarter plane into a semi-circle of unit radius in the lower z-half-plane with the points $z=\pm 1$, as the extremities of the bounding diameter.

4. Thus $u_2=\frac{z}{z^2+1}(x_1-x_2)+\frac{x_1+x_2}{2}$, and $u_1=\frac{w_1}{\pi}\log\frac{w}{q}-w'_1-\nu_1/2$,

$$\therefore \frac{z}{z^2+1}(x_1-x_2)+\frac{x_1+x_2}{2}=\frac{1}{2}\,\frac{\mathfrak{S}'\left(\frac{w_1}{\pi}\log w-\frac{\nu_1}{2}\Big/ w'_1,\ w'_3\right)-\mathfrak{S}'(\nu_1/w'_1,w'_3)}{\mathfrak{S}\left(\frac{w_1}{\pi}\log w-\frac{\nu_1}{2}\Big/ w'_1,\ w'_3\right)-\mathfrak{S}(\nu_1/w'_1,\ w'_3)}+\frac{S_1}{4}$$

$$=\frac{i}{2}\,\frac{\mathfrak{S}'\left(\frac{w_1}{i\pi}\log w+\frac{i\nu_1}{2}\Big/ w_1,\ w_3\right)+\mathfrak{S}'(i\nu_1/w_1,\ w_3)}{-\mathfrak{S}\left(\frac{w_1}{i\pi}\log w+\frac{i\nu_1}{2}\Big/ w_1\ \ w_3\right)+\mathfrak{S}(i\nu_1/w_1,\ w_3)}+\frac{S_1}{4}$$

[Since, on account of homogeneity, $\mathfrak{S}(u/w'_1,\ w'_3)=\mathfrak{S}(u/-iw_3,\ iw_1)=\mathfrak{S}(u/iw_1, iw_3)$, (suppressing the negative sign for a period), $=-\mathfrak{S}(-iu/w_1, -w_3)=-\mathfrak{S}(iu/w_1,\ u_3)$; and $\mathfrak{S}'(u/w'_1,\ w'_3)=-i\mathfrak{S}'(iu/w_1,\ w_3)$],

$$=i\left[\zeta\left(\frac{w_1}{i\pi}\log w+\frac{i\nu_1}{2}\right)-\zeta\left(\frac{w_1}{i\pi}\log w-\frac{i\nu_1}{2}\right)-\zeta(i\nu_1)\right]+\frac{S_1}{4},$$

(the semi-periods being w_1 and w_3).

This would transform the upper semi-couronne of the w-plane into a semi-circle in lower z-half-plane with $z=\pm 1$, as extremities of its bounding diameter.

Correspondance.

To a point $w=e^{i\theta}$ on the unit circle would correspond a certain point x on the real z-axis, given by

$$\frac{x}{x^2+1}(x_1-x_2)+\frac{x_1+x_2}{2}=i\left[\zeta\left(\frac{w_1}{\pi}\theta+\frac{i\nu_1}{2}\right)-\zeta\left(\frac{w_1}{\pi}\theta-\frac{i\nu_1}{2}\right)-\zeta(i\nu_1)\right]+\frac{S_1}{4},$$

and $f(\theta)=F(x)$ say, where x and θ are connected by the above relation,

$$0\leq\theta\leq\pi,\quad |x|\leq 1.$$

Transformation.

$$w=qe^{\frac{\pi}{w_1}u}=qe^{\frac{\pi}{w_1}\left(\frac{\nu_1}{2}+\frac{w_3}{i}+u\right)}=e^{\frac{\pi}{w_1}\left(\frac{\nu_1}{2}+u_1\right)},$$

therefore,

$$\frac{w_1}{\pi}\log w=\frac{\nu_1}{2}+u_1=\frac{\nu_1}{2}+\int_{\infty}^{u_1}\frac{dt}{\sqrt{(t-x_1)(t-x_2)(t-x_3)(t-x_4)}}$$

$$=\frac{\nu_1}{2}+\int_{\infty}^{\frac{x_1+x_2}{2}(x_1-x_2)\frac{z}{z^2+1}}\frac{dt}{\sqrt{(t-x_1)(t-x_2)(t-x_3)(t-x_4)}}$$

$=T(z)$, say, the integration being taken along a direct path.

The function

$$\psi(z)=\frac{i\pi}{w_1^2}\int_0^{\pi} f(\epsilon)\left[\zeta\left(\mathrm{T}(z)-\frac{w_1}{\pi}\epsilon\right)+\zeta\left(\mathrm{T}(z)+\frac{w_1}{\pi}\epsilon\right)-\right.$$

$$\left.-\zeta_3\left(\mathrm{T}(z)-\frac{w_1}{\pi}\epsilon\right)-\zeta_3\left(\mathrm{T}(z)+\frac{w_1}{\pi}\epsilon\right)\right]d\epsilon,$$

(w_1 and $2w_3$ being the semi-periods),

would have its real part equal to $\mathrm{F}(x)$ at a point $z=x$ on the real axis and the normal derivative of its real part would be zero over the semi-circular boundary and is regular within the semi-circular area. This real part is

$$\mathrm{R}\psi(z)=\frac{i\pi}{2w_1^2}\int_0^{\pi} f(\epsilon)\left[\wp'(i\mathfrak{I}\mathrm{T}(z)+2w_3)\left\{\frac{1}{\wp\left(\mathrm{RT}(z)-\frac{w_1}{\pi}\epsilon\right)-\wp(i\mathfrak{I}\mathrm{T}(z)+2w_3)}\right.\right.$$

$$\left.+\frac{1}{\wp\left(\mathrm{RT}(z)+\frac{w_1}{\pi}\epsilon\right)-\wp(i\mathfrak{I}\mathrm{T}(z)+2w_3)}\right\}-\wp'(i\mathfrak{I}\mathrm{T}(z))\left\{\frac{1}{\wp\left(\mathrm{RT}(z)-\frac{w_1}{\pi}\epsilon\right)-\wp(i\mathfrak{I}\mathrm{T}(z))}\right.$$

$$\left.\left.+\frac{1}{\wp\left(\mathrm{RT}(z)+\frac{w_1}{\pi}\epsilon\right)-\wp(i\mathfrak{I}\mathrm{T}(z))}\right\}-\frac{2\wp'(i\mathfrak{I}\mathrm{T}(z))}{\wp(i\mathfrak{I}\mathrm{T}(z))-\wp(2w_3)}\right]d\epsilon,$$

(w_1 and $2w_3$ being the semi-periods of the elliptic functions)

which would be harmonic in the semi-circular area, according to the theory of analytic functions of a complex variable.

UNIVERSITY OF DACCA.

Reference

Villat, H. 1916, *Acta Mathematica*, 40,

ON A GENERALISATION OF REIDEMEISTER'S FIGURE IN A WEB

By

A. C. Choudhury

(Received April 28, 1943)

1. In a previous paper (1942), I have generalised Thomsen's triangle to figures which I have called one-circuit-figures. It has been shown there that these figures give rise to relations in a quasi-group which are usually associative-commutative. But in the geometry of 3-webs there exists another figure, namely Reidemeister's figure, which plays an equally important role. An essential difference between the two figures can easily be seen.

In a Thomsen's triangle (Fig. 1) one can start from the point A and move along ADEBCFA. In going along this path, the lines AB, CD, EF are not traversed at all. These lines were called 'non-essential lines' in my paper noted above. A Thomsen's triangle can also be considered as having been formed from a rectangular strip of paper ABEFDC which has been folded and joined along AB after being given a twist in such a manner that the two edges ADEB and BCFA of the strip are joined into one edge ADEBCFA. The non-essential lines are exactly the creases of the paper. On the other hand, Reidemeister's figure (Fig. 2) may be constructed from a rectangular strip of paper whose edges are ABCD and EFGH, by folding it along DH, CG, BF and by joining along AE without a twist. The lines AE, BF, CG and DH are only creases of the paper and are therefore 'non-essential lines' in the sense of my previous paper. The two edges may be considered as the two 'circuits' of the figure. H. Dibbert (1937) has used this property to generalise Reidemeister's figure to n-webs. The difference between two figures is similar to that between one-sided and two-sided surfaces.

Fig 1.

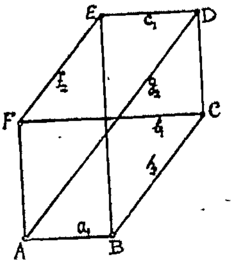
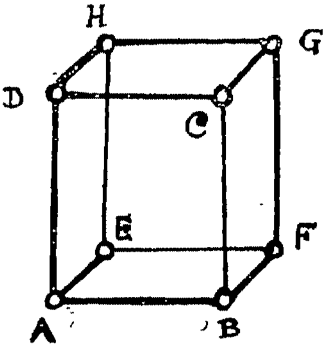


Fig. 2



The same property may be used to generalise Reidemeister's figure to other figures in a 3-web. In this paper, I have studied, besides Reidemeister's figure, the next simple case in which the edges are two hexagons from the point of view of the relations generated in the quasi-group corresponding to the 3-web. It is found that the fact that this figure is closed signifies that a relation $ad=bc$ admits a commutative transformation, though the products ad and bc may not be commutative.

2. Let us consider the relations produced by Reidemeister's figure in a quasi-group. Suppose that the lines AB, EF, DC, HG in the above figure are 1-lines a_1, d_1, g_1, i_1 and the lines AE, BF, CG and DH are 2-lines b_2, c_2, h_2, f_2 respectively. The other four lines are 3-lines only. Then the figure generates the system of relations

$$ba=fg,\ fi=bd,\ ca=hg,\ cd=hi.$$

This system of relations can be reduced to well-known simpler relations. Firstly, let a be the right identity for b, c and let f be the left identity for g, i; then the system reduces to the associative relation

$$(hb)d=h(bd).$$

Next, let a be the right identity for b and c, while b itself is the left identity for a and d. Then $ba=a=b$, and therefore $a^2=a$, *i.e.*, a is an idempotent element. Thus the system of relations reduce to $fg=a$, $(hg).(fi)=fi$. Lastly, let a be the right identity for b, c. Then the relations reduce to $fi=(fg)d$ and $hi=(hg)d$. Further, let there be two elements p, q, such that $p(fi)=i$ and $q(pi)=i$; then the system finally reduces to

$$(p(fg)d)=q((hg)d).$$

It may be noticed here that three triangular relations are equivalent to an associative relation. For, from the relations (*i*) $(ac)\ b=(ab)c$, (*ii*) $ca=ac$, (*iii*) $(ab)c=c(ab)$, the relation $c(ab)=(ca)b$ follows at once. This can also be proved by means of the figures in the corresponding 3-webs (Blaschke und Bol, 1938, p. 38). Therefore, it follows also that the system of relations generated by Reidemeister's figure is equivalent to 3 sets of triangular relations.

Two interesting special cases of Reidemeister's figure were obtained by Bol (1937) by making two non-essential lines of the figure coincident. Let DC and EF belong to the same 1-line and therefore $g_1=d_1$. The system of relations reduces to

$$ba=fd,\ fi=bd,\ ca=hd,\ cd=hi.$$

Let f be the common left identity of d, i and d the common right identity of f, b, h, c. Then $fg=f=g$, and therefore $f^2=f$, which shows that the element f is an idempotent element. The system reduces after some calculation to $((hb)a)b=h((ba)b)$ which is Bol's U_1-relation. We can get another relation between a, b, c under the same condition. For

$$d=ba,\ h=hd=h(ba)=(ca).(ba)$$

and $$c=hi=hb=(ca)b,\ h=ca=((ca)b)a.$$

Thus $$((ca)b)a=(ca).(ba).$$

Again, let the lines CG and AE belong to the same 2-line. Then $h_2=b_2$ and therefore the system of relation is now

$$ba=fg,\ fi=bd,\ ca=bd,\ cd=bi.$$

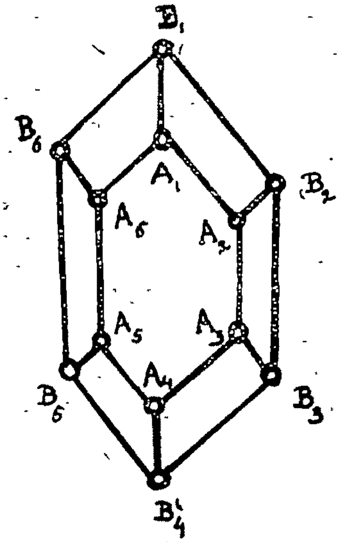
Let b be the common left identity for a, d, g, i. Then $a=fg$, $d=fi$, $g=ca$, $i=cd$, and therefore $a=f(ca)$, $d=f(cd)$, $g=c(fg)$ and $i=c(fi)$. If further a is the right identity of c, then $g=c$, and therefore $c=c(fc)$. Thus

$$i=cd=(c(fc))d=c(f(cd)).$$

The relation $(c(fc))d=c(f(cd))$ is Bol's U_2-relation.

Reidemeister's figure was bounded by two quadrilateral edges. Evidently, one can consider figures bounded by two $2n$-sided polygons. The relations generated are more numerous in these cases. As the next simple case, consider the figure having two hexagonal edges (Fig. 8) which may be called Dibbert's figure.

FIG. 8



Let the lines B_4B_5, A_4A_5, A_1A_2, B_1B_2, B_6A_6, A_3B_3 be 1-lines a_1, b_1, c_1, d_1, g_1, f_1 respectively. Also, suppose that B_4B_3, A_4A_3, A_6A_1, B_6B_1 B_5A_5, A_2B_2 are 2-lines l_2, m_2, n_2, p_2, q_2, r_2 respectively, while the remaining lines of the figure are 3-lines. Then the figure generates in the quasi group the system of six relations

$$la=mb,\ nc=pd,\ rc=mf,\ rd=lf,\ qb=ng,\ qa=pg.$$

If q be the left identity of a, b and r the left identity of c, d, then it is easily found that $l(pg)=m(ng)$ and $n(mf)=p(lf)$. Again if g be the right identity for p, n and f the right identity for m, l, then $(rd)a=(rc)b$ and $(qb)c=(qa)d$. If both these conditions are true, namely that q is the left identity for ab, r is the left identity for cd, g is the right identity for p, n and f is the right identity for m, l, then it is found that

$$da=cb \text{ and } bc=ad.$$

Thus the closedness property of a Dibbert's figure is that a relation $ad=bc$ admits of a commutative transformation although the products ad, bc may not be commutative.

Conversely if in a quasi-group, a relation $ad=bc$ admits of a commutative transformation, *i.e.*, $da=cb$ is also true, then the lines a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 and their common left and right identity elements (if these exist) will enclose a Dibbert's figure. For example, let p_1, q_2 be the common right and left identity elements of a, b and r_1, s_2 be those of c, d. Then six relations $a_2b_1=b_2c_1$, $d_2a_1=c_2b_1$, $a_2p_1=q_2a_1$, $b_2p_1=q_2b_1$, $c_2r_1=s_2c_1$, $d_2r_1=s_2d_1$ are obtained. These six relations between twelve elements a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, p_1, q_2, r_1, s_2, will define a Dibbert's figure. Evidently, one gets other Dibbert's figures from the same relation $ad=bc$ by taking identity elements of other pairs, say a, c and b, d.

Consider now special cases of the figure. If the two lines A_3B_3 and A_6B_6 coincide, the line may be called the 1-diagonal of the figure. In this case, $f_1=g_1$ and therefore the above relations reduce to

$$l(pf)=m(nf), \quad p(lf)=n(mf).$$

Again if the lines A_5B_5 and A_2B_2 belong to the same 2-line, this line may be called the 2-diagonal of the figure. Here also $r_2=q_2$ and the second system of relations deduced above gives

$$(qd)a=(qc)b, \quad (qa)d=(qb)c.$$

If f_1 and r_2 are the right and left identity of elements, then the Dibbert's figure with the two diagonals signifies that $da=cb$ when $ad=bc$. Thus in a 3-web corresponding to a quasi-group with unique left and right identity elements, this is the figure whose closedness property is that a relation is commutative although the products themselves are not commutative. Now let the two remaining lines A_1B_1 and A_4B_4 belong to the same 3-line. This 3-line may be called the 3-diagonal of the figure. Then these products are equal, *i.e.*, $ad=bc=da=cb$. Thus the existence of the 3-diagonal gives that the products ad and bc are commutative. If all these three diagonals met in a point, then we get a further result that $ad=bc=f$ where $f^2=f$, *i.e.*, the products ad, bc are equal, commutative and idempotent.

References

Blaschke und Bol, 1938, *Geometrie der Gewebe*, Berlin, 32-41.
Bol, G., 1937, *Math Ann.*, 114, 414.
Choudhury, A. C., 1942, *Bull Cal Math. Soc.*, 34. 93-98.
Dibbert, H., 1937, *Math. Ges. Hamburg*, 7, 323-354.

ON EXPANSION IN EIGENFUNCTIONS OF BOUNDARY VALUE PROBLEMS—VI

BY

S. MINAKSHISUNDARAM

(Communicated by the Secretary—Received May 13, 1943)

§1. In 5 previous papers with the same title, I have studied the problem of expanding an arbitrary function of two variables in a series of eigenfunctions. All the results, so far obtained, could be extended to a space of m-dimensions, $m \geq 2$, and in this note I point out the forms that these results take in the space of m-dimensions. Proofs are analogons and only brief indications are given, where necessary.

Consider the equation

$$\triangle w + \mu w = \sum_{i=1}^{m} \frac{\partial^2 w}{\partial x_i^2} + \mu w = 0 \qquad \ldots \quad (1)$$

with the boundary condition

$$w(x) = w(x_1, x_2, \ldots x_m) = 0 \text{ on C}, \qquad \ldots \quad (2)$$

where C is the boundary of an m-dimensional domain D. We assume D and C to be sufficiently regular such that (1) and (2) have solutions for an infinite number of eigenvalues μ_n (arranged in increasing order of magnitude) with the corresponding eigenfunction w_n. We shall state some of the fundamental properties of eigenfunction and eigenvalues. These properties are known for $m \leq 3$ and could easily be extended for $m \geq 3$ as well.

$$\lim_{n=\infty} \frac{\mu_n}{n^{\frac{2}{m}}} \geq 0 \qquad \ldots \quad (3)$$

so that the series $\Sigma \mu_n^{-\gamma}$ converges for $\gamma > \frac{m}{2}$. We write as usual $\lambda = \mu^{\frac{1}{2}}$, $\lambda_n = \mu_n^{\frac{1}{2}}$.

$$w(x) \mathrm{J}_{\frac{m-2}{2}}(\lambda r) = \frac{1}{\tau_m r^{m-1}} \int w(y)\, d\sigma_{m-1} \qquad \ldots \quad (4)$$

where $w(x)$ is a solution of (1), $\sum^{m} (y_i - x_i)^2 = r^2$, τ_m is the 'area' of the spherical surface of unit radius and $d\sigma_{m-1}$ is the surface element of a sphere. From (4) it is easily proved as in III that

$$w_n(x) = O(\lambda_n^{\frac{1}{2}}) = O(\mu_n^{\frac{1}{4}}) = O(n^{\frac{1}{2m}}) \qquad \ldots \quad (5)$$

so that the series

$$\Sigma \frac{w_n(x) w_n(y)}{\mu_n^{\frac{m+1}{2}+\epsilon}} \qquad \ldots \quad (6)$$

is absolutely convergent for x and y in D.

§2. The results (3) to (6) are, in general, sufficient to prove most of the important results. Let $f(x)=f(x_1 \dots x_m)$ be a function integrable L^2 in D and let

$$f(x) \sim \Sigma a_n w_n(x) \qquad \dots \quad (7)$$

$$f_p(r, x)=\frac{p}{m\tau_m r^m}\int f(y)\left(1-\frac{t^2}{r^2}\right)^{p-1} dy \qquad \dots \quad (8)$$

where $p>0$, $\overset{m}{\Sigma}(y_i-x_i)^2=t^2 \leq r^2$ and dy is m-dimensional volume element. We have now the following results:

THEOREM 1.—*If the Fourier coefficients of $f(x)$ fulfill the condition*

$$\Sigma \mu_n^{\frac{m+1}{2}+\varepsilon} a_n^2 < \infty \qquad \dots \quad (9)$$

then its Fourier series is absolutely convergent at every point of D, in fact uniformly in every domain contained in D.

For if
$$r=\frac{m+1}{2}+\varepsilon$$

$$\Sigma\, |a_n w_n| \leq \left\{ \Sigma \mu_n^r a_n^2 \Sigma \frac{w_n^2}{\mu_n^r} \right\}^{\frac{1}{2}}.$$

THEOREM 2.—*If $m \leq 3$ and $\Sigma \mu_n^{\frac{m}{2}} a_n^2 < \infty$ then the Fourier series of $f(x)$ converges at a point x in D if and only if $\lim\limits_{r=0} f_1(r)$ exists and the sum of the series is equal to this limit.*

The sufficiency of the condition was proved in II; and the necessity follows as in theorem D of III for $m=3$ as well.

Corresponding to theorems A and B of III we have the following:

Theorem 3.—If $f_p(r, x) \longrightarrow s$ as $r \longrightarrow 0$ at a point x in D, then $\Sigma\, a_n w_n$ is summable $(R\mu, k)$ for $k > max\left(\frac{m+1}{2},\ p+\frac{m-1}{2}\right)$.

This is proved as in III on observing that the series $\Sigma\, a_n w_n \alpha_q(\lambda_n r)$ is absolutely convergent for $q \geq \frac{m}{2}$.

Theorem 4.—If the Fourier series of $f(x)$ is summable $(R\mu,\ k)$ to the sum s, then $f_p(r, x) \longrightarrow s$ as $r \longrightarrow 0$ provided $p \geq max.\left(\frac{m}{2}, k-\frac{m-3}{2}\right)$. If $m \leq 3$ we can take $p \geq max.\left(\frac{m-1}{2}, k-\frac{m-3}{2}\right)$.

This is also proved as in II with the aid of the following lemma which is the analogue of lemma 7 in III.

Lemma.—If $f(x) \varepsilon L^2$ and $f(x) \sim \Sigma\, a_n w_n$ then

$$C(w)=\sum^{\lambda_n \leqq w} a_n w_n = O(w^{\gamma}).$$

where

$$\gamma = \begin{cases} \dfrac{m}{2} \text{ for } m \leq 8 \\[2ex] \dfrac{m+1}{2} \text{ for } m \geq 8 \end{cases}$$

If $m > 8$

$$| C(w) | = | \Sigma\, a_n w_n | \leq \{ \Sigma\, a_n^2 \,\Sigma\, w_n^2 \}^{\frac{1}{2}}$$

$$= O\{ (\Sigma\, \mu_n^{\frac{1}{2}})^{\frac{1}{2}} \} = O\{ (\Sigma\, n^{\frac{1}{m}})^{\frac{1}{2}} \}$$

$$= O(n^{\frac{m+1}{2m}}) = O(w^{\frac{m+1}{2}}).$$

If $m \leq 8$, the lemma is proved as in III from the formula

$$\sum^{n} a_{\nu} w_{\nu} = s_n = M_r[s_n] + \int \psi_r \Delta s_n dv_m$$

using the relation

$$H[\Psi_r] \leq \begin{cases} \tau r^2 & \text{for } m=2 \\ \tau r & \text{for } m=8. \end{cases}$$

There is no difficulty in proving the closure and completeness of these eigenfunctions. The method used in IV can be adopted here with the necessary changes. Also the theorems on the classes of functions and their Fourier series hold good here also, with the difference that $k > \frac{m+1}{2}$.

§8. We shall finally prove a theorem with the aid of which the behaviour of of Fourier series of functions belonging to Lip (α, 2) can be studied even as we have done in V.

Theorem 5.—If $f(x) \in L^2$ *and* $\Sigma\, \mu_n^{\gamma} a_n^2 < \infty$, $\gamma > \frac{1}{2}$ *then*

$$(i)\ \Sigma\, | a_n w_n |^{\beta} < \infty \ \text{for}\ \beta > \frac{2m}{2\gamma - 1 + m},$$

$$(ii)\ \Sigma\, \mu_n^{\delta} | a_n w_n | < \infty \ \text{for}\ \delta < \frac{\gamma}{2} - \frac{m+1}{4}.$$

(*i*) For, if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$

$$\Sigma\, | a_n w_n |^{\beta} \leq \left\{ \Sigma\, \mu_n^{\delta p} | a_n |^{\beta p} \right\}^{\frac{1}{p}} \left\{ \Sigma\, \frac{w_n^{\beta q}}{\mu_n^{\delta q}} \right\}^{\frac{1}{q}}$$

$$\leq k \left\{ \Sigma\, \mu_n^{\gamma} a_n^2 \right\}^{\frac{1}{p}} \left\{ \Sigma\, \frac{1}{\mu_n^{(\delta - \frac{\beta}{4})q}} \right\}^{\frac{1}{q}},$$

where we choose $\beta p = 2$ or $p = \frac{2}{\beta} \geq 1$, and $\delta p = \gamma$ or $\delta = \frac{\gamma}{p} = \frac{\gamma\beta}{2}$. Substituting the values of δ, and q and using the asymptotic expression for μ_n, we have

$$\Sigma \mid a_n w_n \mid^{\beta} \leq k \left\{ \Sigma \mu_n^{\gamma} a_n^2 \right\}^{\frac{1}{p}} \left\{ \Sigma \frac{1}{n^{\frac{\beta q}{m}(\gamma-\frac{1}{2})}} \right\}^{\frac{1}{q}}$$

$$< \infty,$$

provided $$\frac{\beta q}{m}(\gamma-\tfrac{1}{2}) > 1,$$

that is, $$\frac{2\beta}{m(2-\beta)}(\gamma-\tfrac{1}{2}) > 1.$$

or $$\beta > \frac{2m}{2\gamma-1+m}.$$

The assumption $p=\frac{2}{\beta} \geq 1$ requires $\gamma > \frac{1}{2}$.

(*ii*) $$\Sigma \mu_n^{\delta} \mid a_n w_n \mid \leq \left\{ \Sigma \mu_n^{\gamma} a_n^2 \; \Sigma \mu_n^{2\delta-\gamma} w_n^2 \right\}^{\frac{1}{2}}$$

$$= O(\Sigma \mu_n^{2\delta-\gamma+\frac{1}{2}}) = O(1),$$

if $$\gamma - 2\delta - \tfrac{1}{2} > \frac{m}{2} \text{ or } \delta < \frac{\gamma}{2} - \frac{m+1}{4}.$$

ANDHRA UNIVERSITY,
GUNTUR.

References

I—1940, *Proc. Ind. Acad. Sc.* A., **12**, 462-65.
II—1941, *Jour. Ind. Math. Soc.*, **5**, 103-108.
III—1942, *Jour. Ind. Math. Soc.*, **6**, 153-167.
IV—1943, ,, ,, **7**, 17-24.
V—1943, ,, ,, (in the press).

CALCUTTA

BULLETIN

OF THE

CALCUTTA MATHEMATICAL SOCIETY

A limited number of back Volumes of the Bulletin of the Calcutta Mathematical Society will be sold at reduced prices.

For particulars write to :—

THE SECRETARY, CALCUTTA MATHEMATICAL SOCIETY,
92, Upper Circular Road, Calcutta.

PRINTED AND PUBLISHED BY BHUPENDRALAL BANERJEE AT THE CALCUTTA
UNIVERSITY PRESS, 48, HAZRA ROAD, BALLYGUNGE, CALCUTTA

BULLETIN

OF THE

CALCUTTA

MATHEMATICAL SOCIETY

CALCUTTA UNIVERSITY CENTRAL LIBRARY

BOARD OF EDITORS

N. M. BASU, D.Sc. F. W. LEVI, DR.PHIL.NAT. C. V. H. RAO, M.A.
N. R. SEN, D.Sc., PH.D. M. R. SIDDIQI, M.A., PH.D.
S. GHOSH, D.Sc. (*Secretary*)

VOL. 35 SEPTEMBER, 1943 No. 3

CONTENTS

PAGE

13. A Case of Polytropic Equilibrium Configuration with variable Angular Velocity—By Sunil Kumar Roy ... 85

14. On a Measure of Divergence between two Statistical Populations defined by their Probability Distributions —By A. Bhattacharyya 99

15. On S-groups—By M. A. Jabber 111

CALCUTTA UNIVERSITY PRESS
1944

CALCUTTA UNIVERSITY CENTRAL LIBRARY

A CASE OF POLYTROPIC EQUILIBRIUM CONFIGURATION WITH VARIABLE ANGULAR VELOCITY

By

Sunil Kumar Roy

(Communicated by Prof. A. C. Banerji—Received, March 25, 1943)

1. In the solution of the problem of the equilibrium of polytropes distorted by a small uniform rotation given by Milne (1923), von Zeipel (1924), Chandrasekhar (1933), and others, Kopal (1939) has pointed out an easier alternative to the method of numerical integration in the special case when the polytropic index is unity, and has given the solution in finite terms in terms of the Bessel functions. In this paper we shall discuss the corresponding problem for variable rotation. The variation is supposed to be spatial only and not temporal, so as to be equivalent to the superimposing of a system of variable forces, constant in time. The spatial dependence allows us the simplification of using Poincare's result for steady rotation with variable angular velocity.

The equations of rotational equilibrium, with the usual notations, are

$$-\omega^2 r\sin^2\theta=\frac{\partial V}{\partial r}-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \dots \quad (1)$$

and

$$-\omega^2 r\sin\theta\cos\theta=\frac{\partial V}{r\partial\theta}-\frac{1}{\rho}\frac{\partial p}{r\partial\theta}. \qquad \dots \quad (2)$$

For the polytrope with index n, we write as usual

$$\rho=\lambda\Theta^n,\ p=K\rho^{1+\frac{1}{n}}=K\lambda^{1+\frac{1}{n}}\Theta^{n+1},\ \frac{dp}{\rho}=K(n+1)\lambda^{\frac{1}{n}}\,d\Theta=R\,d\Theta. \qquad \dots \quad (3)$$

Then ω^2 being supposed to depend on the distance from the axis, (1) and (2) give $V-R\Theta=-\int\omega^2 r\sin^2\theta\,dr=-\int\omega^2 r^2\sin\theta\cos\theta\,d\theta=$ a function of $r\sin\theta$.

Thus we can write $\qquad V-R\Theta=-f(r^2(1-\mu^2))+\text{const.} \qquad \dots \quad (4)$

where $\qquad \mu=\cos\theta$, and $f'(x)=\frac{1}{2}\omega^2(x). \qquad \dots \quad (5)$

Substituting in Poisson's equation, and changing from r to ξ by writing

$$r^2=\frac{R}{4\pi\gamma\lambda}\xi^2=\frac{K\lambda^{\frac{1}{n}-1}(n+1)}{4\pi\gamma}\,\xi^2$$

we find

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Theta}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta}{\partial\mu}\right)=-\Theta^n+\frac{F}{4\pi\gamma\lambda},$$

where

$$F=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial f}{\partial\mu}\right).$$

For $n=1$, this equation is a linear equation in Θ. Hence if f is a sum of a set of functions, *i.e.*, $f=\sum_s f_s$ then F is also of the form $\sum_s F_s$ and we can write $\Theta=\sum_s \Theta_s$

where
$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Theta_s}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta_s}{\partial\mu}\right)=-\Theta_s+\frac{F_s}{4\pi\gamma\lambda}, \qquad \dots (6)$$

$$F_s=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f_s}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial f_s}{\partial\mu}\right). \qquad \dots (7)$$

The simplest form for f_s follows from the consideration that since f is a function of $r\sin\theta$, f can be expanded in a series of powers of $r\sin\theta$, and we can take

$$f_s=r^s(1-\mu^2)^{s/2}$$

corresponding to which
$$(\omega^2)_s=sr^{s-2}(1-\mu^2)^{\frac{s-2}{2}}.$$

Expanding f_s in the form $r^s\sum_m A_{ms}P_m(\mu)$, where $A_{ms}=\frac{2m+1}{2}\int_{-1}^{+1}(1-\mu^2)^{s/2}P_m(\mu)d\mu$

and substituting in (6) we get

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Theta_s}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta_s}{\partial\mu}\right)+\Theta_s=\sum_m B_{ms}\xi^{s-2}P_m(\mu)$$

where B_{ms} are determinate constants given by

$$B_{ms}=\left(\frac{R}{4\pi\gamma\lambda}\right)^{\frac{s-2}{2}}\frac{s(s+1)-m(m+1)}{4\pi\gamma\lambda}A_{ms}.$$

To solve this equation as usual we put $\Theta_s=\sum_j \psi_j(\xi)P_j(\mu)$ where the arbitrary constant usually given as the coefficient of $\psi_j(\xi)$ is supposed here to be included in the function $\psi_j(\xi)$ itself. Separating the coefficients of $P_j(\mu)$ and equating them to zero for every value of j, we get the system of equations

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right)+\left(1-\frac{j(j+1)}{\xi^2}\right)\psi_j=B_{js}\xi^{s-2}. \qquad \dots (8)$$

Now the solution of the equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right)+\left(1-\frac{j(j+1)}{\xi^2}\right)\psi_j=0$$

is given by the associated Emden functions for the index $n=1$, which is given by Kopal in finite terms in the form

$$\psi_j(\xi)=2^j\sqrt{\frac{2}{\xi}}\,\Pi(j+\tfrac{1}{2})J_{j+\frac{1}{2}}(\xi)=2^{j+\frac{1}{2}}(-1)^j\Pi(j+\tfrac{1}{2})\xi^j\left(\frac{d}{\xi d\xi}\right)^j\left(\frac{\sin\xi}{\xi}\right)\sqrt{\frac{2}{\pi}}.$$

Hence, neglecting the constant required for normalisaiton, and including an arbitrary coefficient, we can write $\psi_j(\xi)=a_j\phi_j(\xi)$, where

$$\phi_j(\xi)=\xi^j\left(\frac{d}{\xi d\xi}\right)^j\left(\frac{\sin\xi}{\xi}\right)=(-1)^j\xi^{-\frac{1}{2}}J_{j+\frac{1}{2}}(\xi)$$

Now the complete solution of equation (8), following the usual rules of the solution of linear differential equations of the 2nd order, is

$$\psi_j = a_j(-1)^j \xi^{-\frac{1}{2}} J_{j+\frac{1}{2}}(\xi) + b_j(-1)^j \xi^{-\frac{1}{2}} J_{j+\frac{1}{2}}(\xi) \int^{\xi} \frac{1}{x\{J_{j+\frac{1}{2}}(x)\}^2}\, dx$$

$$+ (-1)^j \xi^{-\frac{1}{2}} J_{j+\frac{1}{2}}(\xi) \int^{\xi} \frac{dx}{x[J_{j+\frac{1}{2}}(x)]^2} \int^{x} B_{j,s}\, y^s (-1)^j y^{-\frac{1}{2}} J_{j+\frac{1}{2}}(y)\, dy \;\ldots \quad (9)$$

where a_j and b_j are arbitrary constants, the latter for our purpose being zero. It can be seen that the particular integral here can be put in the form

$$\phi_j(\xi) \int^{\xi} \frac{\chi_{j,s}(x)\,dx}{x^2 \phi_j^{\,2}(x)}, \quad \text{where } \chi_{j,s}(x) = \int^{x} (-1)^j y^{s+j} \left(\frac{d}{y\,dy}\right)^j \left(\frac{\sin y}{y}\right) dy \;\ldots \quad (10)$$

and $\chi_{j,s}(x)$ satisfies the relation

$$\chi_{j,s}(x) = (s+j-1)\chi_{j-1,s-1}(x) - x^s \phi_{j-1}(x). \quad \ldots \quad (11)$$

If $j \lessgtr s$ we use (11) up to the function $\chi_{0,s-j}(x)$ where

$$\chi_{0,s-j}(x) = \int^{x} y^{s-j-1} \sin y\, dy$$

which equals
$$\{x^{s-j-1} \sin\left(x - \frac{\pi}{2}\right) - (s-j-1)\sin(x-\pi).x^{s-j-2} + \ldots$$

$$+ (-1)^k (s-j-1)(s-j-2)\ldots(s-j-k) \sin\left(x - \frac{k+1}{2}\pi\right) x^{s-j-k-1} + \ldots$$

$$+ (-1)^{s-j-1}(s-j-1)!\ \sin\left(x - \frac{s-j}{2}\pi\right)\}. \quad \ldots \quad (11\text{A})$$

If $j \gtrless s$, we shall have to use (11) up to the function $\chi_{j-s,0}(x)$ where

$$\chi_{j-s,0}(x) = \int^{x} (-1)^{j-s} y^{j-s} \left(\frac{d}{y\,dy}\right)^{j-s} \left(\frac{\sin y}{y}\right) dy = (-1)^{j-s} \int^{x} \phi_{j-s}(y)\, dy \ldots \quad (11\text{B})$$

The particular integral in (9) at first sight may appear to give an infinity at $\xi = 0$. On closer examination however the apprehension would appear to be false.

Thus our complete solution for eq. (8) is

$$\psi_j(\xi) = a_j \sqrt{\frac{2}{\pi}}\, 2^{j+\frac{1}{2}} (-1)^j \Pi(j+\tfrac{1}{2}) \phi_j(\xi) + B_{j,s} \phi_j(\xi) \int^{\xi} \frac{\chi_{j,s}(x)\,dx}{x^2 \phi_j^{\,2}(x)} \quad \ldots \quad (12)$$

where a_j is an arbitrary constant, and the other factors have been included, as given by Kopal, to retain normalisation. In the integral involved here, it will be observed that the integrand remains finite at $x=0$, if $s \geqq j+1$, while if $s=j$ we know $B_{j,s}=0$, if $s \leqq j-1$, the integral diverges, but the coefficient of $B_{j,s}$ in (12) remains small due to the presence of the multiplying factor $\phi_j(\xi)$. Hence we can always find out the value of this integral, and so of $\psi_j(\xi)$ at any point.

The expression for the potential can be written down from (4).

2. For the law of variation $(\omega^2)_s = \sum_s C_{s+2} r^s (1-\mu^2)^{s/2}$ from the last section we get

$$V(\xi,\mu) = R\sum_j\left[\sum_s \frac{C_s}{s}\psi_{j,s}(\xi) - \sum_s \frac{C_s}{s}\,\frac{B_{j,s}\xi^s}{s(s+1)-j(j+1)}\right]P_j(\mu), \quad \dots \quad (1)$$

$$\Theta(\xi,\mu) = \sum_j\left[\sum_s \frac{C_s}{s}\psi_{j,s}(\xi)\right]P_j(\mu). \quad \dots \quad (2)$$

It will be directly noticeable that instead of using the arbitrary coefficients in the form $a_{j,s}$, we can put down

$$C'_j = \sum_s \frac{C_s}{s} a_{j,s} \sqrt{\frac{2}{\pi}}\,.\,2^{j+\frac{1}{2}}(-1)^j \Pi(j+\tfrac{1}{2})$$

and write down (1) and (2) in the forms

$$V = R\sum_j\left[C'_j\phi_j(\xi) + \sum_s B_{j,s}\left(\frac{-C_s/s}{s(s+1)-j(j+1)}\,\xi^s + \frac{C_s}{s}\phi_j(\xi)\int^{\xi}\frac{(-1)^j\chi_{j,s}(x)dx}{x^2\phi_j^2(x)}\right)\right]P_j(\mu) \quad \dots \quad (3)$$

$$\Theta = \sum_j\left[C'_j\phi_j(\xi) + \sum_s B_{j,s}\frac{C_s}{s}\phi_j(\xi)\int^{\xi}(-1)^j\frac{\chi_{j,s}(x)dx}{x^2\phi_j^2(x)}\right]P_j(\mu). \quad \dots \quad (4)$$

Here it will be observed that with our assumptions relating to the value of the particular integral at $\xi=0$, the condition of normalisation at $\xi=0$ where $\Theta=1$ gives $C'_0=1$ directly.

For the case $j=0$, for the particular integral here it can be easily seen

$$\phi_0(\xi) = \frac{\sin\xi}{\xi}.$$

$$\chi_{0,s}(x) = -(s-1)!\cos x\left[\frac{x^{s-1}}{(s-1)!} - \frac{x^{s-3}}{(s-3)!} + \dots\right]$$
$$+ (s-1)!\sin x\left[\frac{x^{s-2}}{(s-2)!} - \frac{x^{s-4}}{(s-4)!} + \dots\right]$$

one of the series ending in a term in x, and the other in a constant, thus the last term in the series in the second pair of brackets being $(-1)^{\frac{s-2}{2}}$ or $(-1)^{\frac{s-3}{2}}x$ according as s is even or odd. Observing that* there is an additional term in $\chi_{0,s}$ if s is odd, *vis.*, $(-1)^s\ (s-1)!\sin\frac{s\pi}{2}$ we find that on integration, consistent with our intention to make the particular integral vanish at $\xi=0$, we can write

$$\phi_0(\xi)\int^{\xi}\frac{\chi_{0,s}(x)dx}{x^2\phi_0^2(x)}$$

---

* The expression (11A) for $\chi_{0,s-j}(x)$ vanishes for $x=0$, if $s-j$ is even. When $s-j$ is odd, to make $\chi_{0,s-j}(x)$ vanish at $x=0$, we must add a term

$$(-1)^{s-j}(s-j-1)!\,\sin\frac{s-j}{2}\pi$$

in the form

$$(-1)^{s/2}\frac{(s-1)!}{\xi}\left\{\sin\xi-\left(\xi-\frac{\xi^3}{3!}+\ldots+(-1)^{\frac{s-2}{2}}\frac{\xi^{s-1}}{(s-1)!}\right)\right\}\text{ if } s \text{ is even}$$

$$\ldots \quad (5)$$

$$\text{and}\quad(-1)^{\frac{s+1}{2}}\frac{(s-1)!}{\xi}\left\{\cos\xi-\left(1-\frac{\xi^2}{2!}+\ldots+(-1)^{\frac{s-1}{2}}\frac{\xi^{s-1}}{(s-1)!}\right)\right\}\text{ if } s \text{ is odd}$$

This is a good exemplification of a particular case of our analysis of § 1. These expressions for the particular integral can be verified, by integration in series, which in this particular case is rather simple as can be directly seen. For the case $s=2$, we get the familiar result that the particular integral for the constant uniform rotation is

$$-\frac{B_{02}}{\xi}\{\sin\xi-\xi\}=B_{02}\left(1-\frac{\sin\xi}{\xi}\right)=B_{02}\{1-\phi_0(\xi)\}.$$

When the rotation is small, the determination of the arbitrary coefficients C'_j follows standard methods. With the usual treatment, comparing the internal and external potentials and their gradients at $\xi=\xi_1$ $(=\pi$ in our case) it can be deduced that

$$C'_j\{(j+1)\phi_j(\xi_1)+\xi_1\phi'_j(\xi_1)\}=-\sum_s B_{j,s}\left\{-\frac{C_s}{s}\left[\Big((j+1)\phi_j(\xi_1)+\xi_1\phi'_j(\xi_1)\Big)\right.\right.$$

$$\left.\left.\times\int^{\xi_1}\frac{(-1)^j\chi_{j,s}(x)dx}{x^2\phi_j^{\,2}(x)}+\frac{\chi_{j,s}(\xi_1).(-1)^j}{\xi_1\phi_j(\xi_1)}\right]+\frac{C_s\left(\frac{j+1}{s}+1\right)\xi_1^{\,s}}{s(s+1)-j(j+1)}\right\}\ldots\quad(6)$$

$j \geqq 1$. We have already pointed that $C'_0=1$.

3. Characteristic constants of the rotating polytrope.—We know that $\phi_0(\xi)$ is the Emden function $\theta_1(\xi)=\frac{\sin\xi}{\xi}$ with index unity. Hence $\phi_0(\xi)=\phi_0(\pi)=0$. Suppose that the boundary is given to the 1st order of small quantities of the order of the square of the angular velocity by $\xi=\xi_1+\Delta\xi_1=\pi+\Delta\xi_1$. Substituting in the equation $\Theta(\xi_1+\Delta\xi_1,\,\mu)=0$ we get

$$C'_0\Delta\xi_1.\Theta'_1(\xi_1)+\sum_j\left[C'_j\phi_j(\xi_1)+\sum_s B_{j,s}\frac{C_s}{s}\phi_j(\xi_1)\int^{\xi_1}\frac{(-1)^j\chi_{j,s}(x)dx}{x^2\phi_j^{\,2}(x)}\right]P_j(\mu)=0$$

C'_0 being of course unity. Hence noting that $\Theta'_1(\xi_1)$ is negative, the boundary is given by

$$\xi=\xi_1+\frac{1}{|\Theta'_1(\xi_1)|}\sum_j\Big[\sum_s B_{j,s}\frac{C_s}{s}\phi_j(\xi_1)\int^{\xi_1}(-1)^j\frac{\chi_{j,s}(x)dx}{x^2\phi_j^{\,2}(x)}+C'_j\phi_j(\xi_1)\Big]P_j(\mu)\ \ldots\quad(1)$$

where ξ_1 is of course $=\pi$, and the C'_j's are known from eq. (6), § 2.

Thus the distortion of the star due to rotation consists of a series of harmonic terms, together with a uniform expansion, which as a fraction of the radius of the star, is given by the expression

$$\frac{\sum_s B_{0s}\frac{C_s}{s}I_s}{\xi_1|\Theta'_1(\xi_1)|}\quad\ldots\quad(2)$$

where the value of I_s from eq. 5, from the previous section, is given by

$$(-1)^{s/2}\frac{(s-1)!}{\pi}\left\{-\left(\pi-\frac{\pi^3}{3!}+\ldots+(-1)^{\frac{s-2}{2}}\frac{\pi^{s-1}}{(s-1)!}\right)\right\}\text{ if } s \text{ is even} \qquad \ldots (3)$$

and $$(-1)^{\frac{s-2}{2}}\frac{(s-1)!}{\pi}\left\{-1-\left(1-\frac{\pi^2}{2!}+\ldots+(-1)^{\frac{s-2}{2}}\frac{\pi^{s-1}}{(s-1)!}\right)\right\}\text{ if } s \text{ is odd.}$$

Also we find $A_{0s}=\frac{1}{2}B\left(\frac{s}{2}+1,\frac{1}{2}\right)=2^s\frac{\{(s/2)!\}^2}{(s+1)!}$ or $\frac{(s+1)!}{2^{s+1}[\{(s+1)/2\}!]^2}\,\frac{\pi}{2}$

according as s is even or odd.

Hence we get $$B_{0s}=\left(\frac{R}{4\pi\gamma\lambda}\right)^{\frac{s-2}{2}}\frac{s(s+1)}{4\pi\gamma\lambda}2^s\frac{\{(s/2)!\}^2}{(s+1)!}\text{ if } s \text{ is even}$$

$$\left(\frac{R}{4\pi\gamma\lambda}\right)^{\frac{s-2}{2}}\frac{s(s+1)}{4\pi\gamma\lambda}\,\frac{(s+1)!}{2^{s+1}[\{(s+1)/2\}!]^2}\,\frac{\pi}{2}\text{ if } s \text{ is odd} \qquad \ldots (4)$$

Thus the proportionate expansion (2) can be put in the form

$$\frac{1}{\xi_1|\Theta'_1(\xi_1)|}\sum_{s\text{ even}}\frac{C_s}{s}\left(\frac{R}{4\pi\gamma\lambda}\right)^{\frac{s-2}{2}}\frac{s(s+1)}{4\pi\gamma\lambda}2^s\frac{\{(s/2)!\}^2}{(s+1)!}$$
$$\times(-1)^{s/2}\frac{(s-1)!}{\pi}\left\{-\left(\pi-\frac{\pi^3}{3!}+\ldots+(-1)^{\frac{s-2}{2}}\frac{\pi^{s-1}}{(s-1)!}\right)\right\}$$
$$+\frac{1}{\xi_1|\Theta'_1(\xi_1)|}\sum_{s.\text{ odd}}\frac{C_s}{s}\left(\frac{R}{4\pi\gamma\lambda}\right)^{\frac{s-2}{2}}\frac{s(s+1)}{4\pi\gamma\lambda}\,\frac{(s+1)!}{2^{s+1}[\{(s+1)/2\}!]^2}\frac{\pi}{2}$$
$$\times(-1)^{\frac{s+2}{2}}\frac{(s-1)!}{\pi}\left\{-1-\left(1-\frac{\pi^2}{2!}+\ldots+(-1)^{\frac{s-1}{2}}\frac{\pi^{s-1}}{(s-1)!}\right)\right\} \qquad \ldots (5)$$

The expression for the difference between the equatorial and the polar distortions can be easily written down.

The volume of the star is given by the integral $-\iiint r^2dr\,d\mu\,d\phi$ taken through the whole star. Knowing the equation to the boundary from (1) and noting that $\int_{-1}^{+1}P_j(\mu)d\mu=0$ for all j except $j=0$, we find

$$\text{Volume}=\left\{\frac{K\lambda^{\frac{1}{n}-1}(n+1)}{4\pi\gamma}\right\}^{\frac{3}{2}}\int_{-1}^{+1}\frac{2\pi(\xi_{\text{Boundary}})^3}{2}d\mu$$
$$=\frac{4\pi}{3}\xi_1^3\left\{\frac{K\lambda^{\frac{1}{n}-1}(n+1)}{4\pi\gamma}\right\}^{\frac{3}{2}}\left\{1+\frac{3}{\xi_1|\Theta'_1(\xi_1)|}\sum_s B_{0s}\frac{C_s}{s}I_s\right\} \qquad \ldots (6)$$

where B_{0s} and I_s have the values given in eqs. (4) and (2), so that the addition to the term 1 inside the last pair of curled brackets in this is simply thrice the quantity set down in (5).

The mass of the star is obtained in the form

$$\text{Mass} = -2\pi\iint \rho r^2 d\gamma d\mu = -2\pi\lambda\left\{\frac{K\lambda^{\frac{1}{n}-1}(n+1)}{4\pi\gamma}\right\}^{\frac{3}{2}}\iint \Theta^n\xi^2 d\xi d\mu \qquad n=1$$

Substituting for $\Theta\xi^2$ from eq. (6) § 1 we get

$$\text{Mass} = 2\pi\lambda\left\{\frac{K\lambda^{\frac{1}{n}-1}(n+1)}{4\pi\lambda}\right\}^{\frac{3}{2}}\int_{-1}^{+1}\int_0^{\xi_1+\Delta\xi_1}\left\{-\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Theta}{\partial\xi}\right)\right.$$

$$\left.-\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta}{\partial\mu}\right)+\frac{F\xi^2}{4\pi\gamma\lambda}\right\}d\mu d\xi.$$

Now over the range $\pi \leqq \xi \leqq \pi+\Delta\xi_1$ the integrand is a small quantity of at least the order of the square of the angular velocity, while the region of integration for this range of values of ξ is also of this order. Hence in neglecting the fourth and higher powers of the angular velocity, we may exclude the region $\pi \leqq \xi \leqq \pi+\Delta\xi_1$ from the range of integration, so that

$$\text{Mass} = 2\pi\lambda\left\{\frac{K\lambda^{\frac{1}{n}-1}(n+1)}{4\pi\gamma}\right\}^{\frac{3}{2}}\int_{-1}^{+1}\int_0^{\xi_1}\sum_s\left[-\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\psi_{0s}}{\partial\xi}\right)\frac{C_s}{s}+B_{0s}\xi^s\frac{C_s}{s}\right]d\mu d\xi$$

$$= 4\pi\lambda\left\{\frac{K\lambda^{\frac{1}{n}-1}(n+1)}{4\pi\gamma}\right\}^{\frac{3}{2}}\sum_s\frac{C_s}{s}\left\{-\xi_1^2\left(\frac{\partial\psi_{0s}}{\partial\xi}\right)_{\xi=\xi_1}+\frac{B_{0s}}{s+1}\xi_1^{s+1}\right\}.$$

Now from § 2 it is easily seen that

$$\sum_s\frac{C_s}{s}\psi_{0s}(\xi) = C'_0\phi_0(\xi)+\sum_{s\text{ even}}\frac{C_s}{s}\,\frac{2^s\{(s/2)!\}^2}{(s+1)!}\left(\frac{R}{4\pi\gamma\lambda}\right)^{\frac{s-2}{2}}\frac{s(s+1)}{4\pi\gamma\lambda}(-1)^{\frac{s-1}{2}}\,\frac{(s-1)!}{\xi}$$

$$\times\left\{\sin\xi-\left(\xi-\frac{\xi^3}{3!}+\ldots+(-1)^{\frac{s-2}{2}}\frac{\xi^{s-1}}{(s-1)!}\right)\right\}$$

$$+\sum_{s\text{ odd}}\frac{C_s}{s}\,\frac{(s+1)!}{2^{s+1}[\{(s+1)/2\}!]^2}\,\frac{\pi}{2}\left(\frac{R}{4\pi\gamma\lambda}\right)^{\frac{s-2}{2}}\frac{s(s+1)}{4\pi\gamma\lambda}(-1)^{\frac{s+1}{2}}\frac{(s-1)!}{\xi}$$

$$\times\left\{\cos\xi-\left(1-\frac{\xi^2}{}+\ldots+(-1)^{\frac{s-1}{2}}\frac{\xi^{s-1}}{(s-1)!}\right)\right\}$$

Hence

$$\sum_s-\xi_1^2\left(\frac{\partial\psi_{0s}}{\partial\xi}\right)_{\xi=\xi_1}\frac{C_s}{s} = -\xi_1^2\phi'_0(\xi_1)+\sum_{s\text{-even}}\frac{C_s}{s}\,\frac{2^s\{(s/2)!\}^2}{(s+1)!}\left(\frac{R}{4\pi\gamma\lambda}\right)^{\frac{s-2}{2}}$$

$$\times\frac{s(s+1)}{4\pi\gamma\lambda}(-1)^{s/2}(s-1)!\left\{\pi+\left(o\,\pi-\frac{2\pi^3}{3!}+\ldots+(-1)^{\frac{s-2}{2}}\frac{(s-2)\pi^{s-1}}{(s-1)!}\right)\right\}$$

$$+\sum_{s\text{-odd}}\frac{C_s}{s}\,\frac{(s+1)!\,\pi/2}{2^{s+1}[\{(s+1)/2\}!]^2}\left(\frac{R}{4\pi\gamma\lambda}\right)^{\frac{s-2}{2}}\frac{s(s+1)}{4\pi\gamma\lambda}(-1)^{\frac{s-1}{2}}$$

$$\times(s-1)!\left\{-1+\left(-1-\frac{1.\pi^2}{2!}+\ldots+(-1)^{\frac{s-1}{2}}\frac{(s-2)\pi^{s-1}}{(s-1)!}\right)\right\}$$

Hence the mass can be put in the form

$$4\pi\lambda\left\{\frac{K\lambda^{\frac{1}{n}-1}(n+1)}{4\pi\gamma}\right\}^{\frac{3}{2}}\xi_1^2\,|\phi'_0(\xi_1)|\left\{1+\frac{1}{\xi_1\,|\Theta'_1(\xi_1)|}\sum_s B_{0s}\frac{C_s}{s}I'_s\right\}\quad\ldots\quad(7)$$

where B_{0s} is given from (4) and I'_s is defined as being equal to

$$(-1)^{s/2}\frac{(s-1)!}{\pi}\left\{\pi+\left(0.\pi-\frac{2\pi^3}{3!}+\ldots+(-1)^{\frac{s-2}{2}}\frac{(s-2)\pi^{s-1}}{(s-1)!}+(-1)^{s/2}\frac{s\pi^{s+1}}{(s+1)!}\right)\right\}$$

if s is even ... (8)

and

$$(-1)^{\frac{s+1}{2}}\frac{(s-1)!}{\pi}\left\{-1+\left(-1-\frac{\pi^2}{2!}+\ldots+(-1)^{\frac{s-1}{2}}\frac{(s-2)\pi^{s-1}}{(s-1)!}+(-1)^{\frac{s+1}{2}}\frac{s\pi^{s+1}}{(s+1)!}\right)\right\}$$

if s is odd.

Thus the mass is known in terms of given constants up to the necessary degree of accuracy. From eqs. (6) and (7), we find that for the star,

$$\frac{\text{Mass}}{\text{Volume}}=\frac{\lambda\,|\phi'_0(\xi_1)|}{\xi_1}\;\frac{1+\dfrac{1}{\xi_1\,|\Theta'_1(\xi_1)|}\sum_s B_{0s}\dfrac{C_s}{s}I'_s}{1+\dfrac{3}{\xi_1\,|\Theta'_1(\xi_1)|}\sum_s B_{0s}\dfrac{C_s}{s}I_s}$$

$$=\frac{\lambda\,|\phi'_1(\xi_1)|}{\xi_1}\left\{1+\frac{1}{\xi_1|\Theta'_1(\xi_1)|}\sum_s\frac{C_s}{s}B_{0s}(I'_s-3I_s)\right\}\quad\ldots\quad(9)$$

up to the degree of accuracy we are working with. Hence noting that $\phi_0(\xi)$ and $\Theta_1(\xi)$ are identical functions, the mean density of the star is given in terms of the central density by the formula

$$\rho_m=\frac{\rho_c\,|\Theta'_1(\xi_1)|}{\xi_1}\left\{1+\frac{1}{\xi_1\,|\Theta'_1(\xi_1)|}\sum_s B_{0s}\frac{C_s}{s}(I'_s-3I_s)\right\}.\quad\ldots\quad(10)$$

4. In § 1, when we obtained the result

$$V-R\Theta=-f(r^2(1-\mu^2))=-f_1(\xi\sqrt{1-\mu^2})$$

our further developments depended upon the supposition that $f_1(\xi\sqrt{1-\mu^2})$ is expansible in powers of $\xi\sqrt{1-\mu^2}$. In the general case we know that this assumption is valid, but among the cases in which we cannot accept the validity of this assumption, we have the important case when

$$V-R\Theta\,\alpha\,-\log\xi\sqrt{1-\mu^2}\quad\text{say}\ =-v\log\xi\sqrt{1-\mu^2}$$

in which case $\omega=\dfrac{\sqrt{v}}{r\sqrt{1-\mu^2}}$. Consideration of this case also yields new results, whose details however are omitted here.

We shall now revert to § 1, and for integration purposes, give the solution of

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right)+\left(1-\frac{j(j+1)}{\xi^2}\right)\psi_j=\mathrm{B}_{j,s}\xi^{s-2}$$

in another form. This equation can be written as

$$\frac{d^2}{d\xi^2}\left(\psi_j\sqrt{\xi}\right)+\frac{1}{\xi}\frac{d}{d\xi}(\psi_j\sqrt{\xi})+\left(1-\frac{(j+\frac{1}{2})^2}{\xi^2}\right)\psi_j\sqrt{\xi}=\mathrm{B}_{j,s}\xi^{s-\frac{3}{2}}.$$

Since $j+\frac{1}{2}$ for our cases is always half-integral, the complete solution of this equation can be written

$$\psi_j(\xi)\sqrt{\xi}=\left\{\mathrm{A}-\frac{\pi}{2\sin(j+\frac{1}{2})\pi}\int^{\xi}\mathrm{B}_{j,s}x^{s-\frac{3}{2}}\mathrm{J}_{j+\frac{1}{2}}(x)x\,dx\right\}\mathrm{J}_{-(j+\frac{1}{2})}(\xi)$$
$$+\left\{\mathrm{B}+\frac{\pi}{2\sin(j+\frac{1}{2})\pi}\int^{\xi}\mathrm{B}_{j,s}x^{s-\frac{3}{2}}\mathrm{J}_{-(j+\frac{1}{2})}(x)x\,dx\right\}\mathrm{J}_{j+\frac{1}{2}}(\xi)\;\ldots\quad(1)$$

following the usual solution by the method of Variation of Parameters. In conformity with what we have been writing in § 1, we put $\mathrm{B}=a_j\,2^{j+\frac{1}{2}}\Pi(j+\frac{1}{2})$, $\mathrm{A}=0$, so that

$$\psi_j(\xi)=a_j2^{j}\sqrt{\frac{2}{\xi}}\,\Pi(j+\tfrac{1}{2})\mathrm{J}_{j+\frac{1}{2}}(\xi)$$
$$-\frac{\pi\mathrm{B}_{j,s}}{2\sin(j+\frac{1}{2})\pi}\int^{\xi}\frac{\mathrm{J}_{-(j+\frac{1}{2})}(\xi)\mathrm{J}_{j+\frac{1}{2}}(x)-\mathrm{J}_{j+\frac{1}{2}}(\xi)\mathrm{J}_{-(j+\frac{1}{2})}(x)}{\sqrt{\xi}}\,x^{s-\frac{1}{2}}dx\;\ldots\quad(2)$$

This can be compared with the solution (9), § 1, which does not involve the Bessel functions of negative orders.

If we extend the definition of $\chi_{j,s}(x)$, given in § 1, to negative values of j, we can use the same recurrence formula as given in that section to deduce the values of $\chi_{j,s}(x)$ and $\chi_{-j-1,s}(x)$ which would serve to complete the solution of eq. 8, § 1 in either the form (9), § 1 involving a quadrature using Bessel functions of positive orders only, or in the form (2) here involving the use of Bessel functions of both positive and negative orders.

5. Consider the application of the foregoing analysis to a particular case. Suppose that

$$\omega^2=\omega_0^2\frac{r^2}{\mathrm{R}_s^2}(1-\mu^2)\qquad\ldots\quad(1)$$

where $\mathrm{R}_s^2=\pi^2\left(\frac{2\mathrm{K}}{4\pi\mathrm{G}}\right)$ *i.e.*, R_s is the radius of the star satisfying the polytropic relationship with index unity. In this case, using the notations of § 1, we write

$$f_4=\frac{\omega_0^2}{4\mathrm{R}_s^2}r^4(1-\mu^2)^2=\frac{\omega_0^2}{4\mathrm{R}_s^2}r^4\left\{\frac{8}{35}\mathrm{P}_4(\mu)-\frac{16}{21}\mathrm{P}_2(\mu)+\frac{8}{15}\right\}$$

so that

$$\mathrm{B}_{j,4}=\frac{1}{4\pi^2}\frac{\omega_0^2}{4\pi\gamma\lambda}\mathrm{A}_j\{20-j(j+1)\}$$

where
$$A_0=\frac{8}{15},\quad A_2=-\frac{16}{21},\quad A_4=\frac{8}{35},\quad A_j=0 \text{ for } j\neq 0,\ 2,\ 4$$

$$B_{04}=\frac{4v}{8\pi^2},\quad B_{24}=-\frac{4v}{8\pi^2},\quad B_{44}=0,\quad B_{j4}=0 \text{ for } j\neq 0,\ 2$$

v in this case being given by $v=\dfrac{\omega_0^2}{2\pi\gamma\lambda}$.

We can therefore write down the equations for $j=0,\ 2$, and proceeding with the calculations indicated, we get

$$\psi_0(\xi)=a_0\frac{\sin\xi}{\xi}-\frac{4v}{8\pi^2}(6-\xi^2),$$

$$\psi_2(\xi)=15a_2\frac{(3-\xi^2)\sin\xi-3\,\xi\cos\xi}{\xi^3}-\frac{4v}{3\pi^2}\xi^2.$$

The actual values of the arbitrary coefficients a_0 a_2 involved herein are deduced as noted previously by comparing the expression for the potential V_{int} obtained from Θ, with the expression for the potential at an external point V_{ext} $(\xi_1\mu)$ of the form $V_{ext}=\left[\dfrac{D_0}{\xi}+\Sigma\dfrac{D_j}{\xi^{j+1}}P_j(\mu)\right]$ on the boundary of the undistorted polytrope.

The working for this simple particular case indicates that though the general methods set forth in § 1 and § 4 can be used for yielding the results in any case, it may probably be easier to solve the problem more directly in simpler particular cases.

In § 1, for $n=1$ we have got the equations

$$V-R\Theta=-f(r^2(1-\mu^2))+\text{constant}$$

and
$$\nabla_\xi^2\Theta+\Theta=\frac{1}{R}\nabla_\xi^2 f$$

the meaning of ∇_ξ^2 being obvious, *viz.*, $\dfrac{1}{\xi^2}\dfrac{\partial}{\partial\xi}\left(\xi^2\dfrac{\partial}{\partial\xi}\right)+\dfrac{1}{\xi^2}\dfrac{\partial}{\partial\mu}\left((1-\mu^2)\dfrac{\partial}{\partial\mu}\right)$.

Now, the complementary functions for this equation are as given by Kopal

viz.,
$$\Theta=\sum_0^\infty a_j\psi_j(\xi)P_j(\mu) \qquad \dots \quad (2)$$

where
$$\psi_j(\xi)=2^j\sqrt{\frac{2}{\xi}}\,\Pi(j+\tfrac{1}{2})J_{j+\frac{1}{2}}(\xi)$$

$$J_{j+\frac{1}{2}}(\xi)=(-1)^j\xi^{j+\frac{1}{2}}\left(\frac{d}{\xi d\xi}\right)^{2j}\left(\frac{\sin\xi}{\xi}\right)$$

To find the particular integral write ∇_ξ^2 in the form $\dfrac{1}{\hat{\omega}}\dfrac{\partial}{\partial\hat{\omega}}\left(\hat{\omega}\dfrac{\partial}{\partial\hat{\omega}}\right)+\dfrac{\partial^2}{\partial z^2}$

where $\hat{\omega}^2=\xi^2(1-\mu^2)=\xi^2 \sin^2\theta$; $z=\xi\cos\theta$. Then the equation becomes

$$\frac{1}{\hat{\omega}}\frac{\partial}{\partial\hat{\omega}}\left(\hat{\omega}\frac{\partial\Theta}{\partial\hat{\omega}}\right)+\frac{\partial^2\Theta}{\partial z^2}+\Theta=\frac{1}{R}\cdot\frac{1}{\hat{\omega}}\frac{d}{d\hat{\omega}}\left(\hat{\omega}\frac{df}{d\hat{\omega}}\right) \qquad \dots \quad (3)$$

since f is independent of z. Now the solution of (3) can be written in the form

$$\Theta=\sum_k E_k e^{kz} J_0(\sqrt{1+k^2}\,\hat{\omega})+\Theta' \qquad \dots \quad (4)$$

where k can be given any set of arbitrary values, and Θ' is given by

$$\frac{1}{\hat{\omega}}\frac{d}{d\hat{\omega}}\left(\hat{\omega}\frac{d\Theta'}{d\hat{\omega}}\right)+\Theta'=\frac{1}{R}\,\frac{1}{\hat{\omega}}\,\frac{d}{d\hat{\omega}}\left(\hat{\omega}\frac{df}{d\hat{\omega}}\right). \qquad \dots \quad (5)$$

The series involving the arbitrary constants, *viz.*, $\sum\limits_k E_k e^{kz} J_0(\sqrt{1+k^2}\,\hat{\omega})$ in (4) and $\sum\limits_j a_j\psi_j(\xi)P_j(\mu)$ in (2) can be replaced one by the other, while the particular integral is given by (5). Suppose now that f can be expanded in powers of $\xi^2(1-\mu^2)$ so that we write f in the form $\sum\limits_s RA_{2s}\hat{\omega}^{2s}$. Then the solution of (5) is found out to be $\Theta'=\sum\limits_s \Theta_{2s}$,

$$\Theta_{2s}=\sum_{k=0}^{k=s-1} 4s^2.A_{2s}\frac{(-1)^{s-k-1}2^{2s-2k-2}\{(s-1)!\}^2}{(k!)^2}\,\hat{\omega}^{2k}. \qquad \dots \quad (6)$$

It can now be expressed in terms of the spherical harmonics, by expanding $\hat{\omega}^{2k}=\xi^{2k}(1-\mu^2)^k$. Thus

$$(1-\mu^2)^k=\sum_{l=0}^{l=k}\frac{k!}{l!(k-l)!}(-1)^l\mu^{2l}$$

Let $$(1-\mu^2)^k=\sum_{n=0}^{n=k} G_{2n}P_{2n}(\mu)$$

Then $$G_{2n}=\sum_{l=n}^{l=k}\int_{-1}^{+1}\frac{k!(-1)^l}{l!(k-l)!}\mu^{2l}\frac{4n+1}{2}P_{2n}(\mu)d\mu$$

Now, $$2^{2n}(2n)!\int_{-1}^{+1}\mu^{2l}P_{2n}(\mu)d\mu=\int_{-1}^{+1}\mu^{2l}\frac{d}{d\mu^{2n}}[(\mu^2-1)^{2n}]d\mu$$

which after $2n$ integration by parts gives

$$2^{2n}(2n)!\int_{-1}^{+1}\mu^{2l}P_{2n}(\mu)d\mu=\frac{(2l)!}{(2l-2n)!}\int_{-1}^{+1}\mu^{2l-2n}(\mu^2-1)^{2n}d\mu$$

$$=\frac{(2l)!}{(2l-2n)!}B(l-n+\tfrac{1}{2},\,2n+1)$$

where $B(m,n)$ is the Euler function of the first kind. Hence we get

$$G_{2n}=\sum_{l=n}^{l=k}\frac{(4n+1)\,(2l)!(l+n)!k!(-1)^l.2^{2n}}{(2l+2n+1)!(l-n)!l!(k-l)!}, \qquad \dots \quad (7)$$

Combining (6) and (7) we can write Θ' in the form

$$\Theta'=\sum_{s} 4s^2 . A_{2s} \sum_{k=0}^{k=s-1} (-1)^{s-k-1} 2^{2s-2k-2} \left[\frac{(s-1)\,!}{k\,!}\right]^2$$

$$\times\, \xi^{2k} \sum_{n=o}^{n=k} \sum_{l=n}^{l=k} \frac{(4n+1)(2l)\,!\,(l+n)\,!\,k\,!\,(-1)^l 2^{2n}}{(2l+2n+1)\,!\,(l-n)\,!\,l\,!\,(k-l)\,!} P_{2n}(\mu) \;\ldots \quad (8)$$

The coefficients of $P_{2n}(\mu)$ can be separated for any prescribed set of values of s. We can verify (8) for simple cases. For example we take the case $2s=2$ corresponding to uniform rotation. Then

$$\Theta'=2^2 . A_2, \text{ a constant.}$$

For the case $2s=4$, which has been treated, we get

$$\Theta'=2^4 A_4 \left\{ -4P_0(\mu)+\xi^2 \left[\left(\frac{1}{1\,!}-\frac{1}{3\,!}\right) P_0(\mu) - \frac{5.2.2.2^2}{5\,!} P_2(\mu) \right] \right\}$$

$$=\frac{2^5 A_4}{3} \left\{ -(6-\xi^2)P_0(\mu) - \xi^2 P_2(\mu) \right\}$$

which can be compared with results given in the beginning noting that corresponding to that case we have here

$$RA_4=\frac{\omega_0^2}{4\pi^2} \frac{R}{4\pi\gamma\lambda} \text{ i.e., } A_4=\frac{v}{8\pi^2} .$$

We shall now write down from (8) the expressions for two more values of s, *viz.*, $s=3$ and $s=4$.

$s=3$:

$$\Theta'=\frac{32}{35} . 3^2 . A_6 \left\{ \left(280-\frac{140}{3}\xi^2+\frac{7}{3}\xi^4\right) P_0(\mu) + \left(\frac{140}{3}\xi^2-\frac{10}{3}\xi^6\right) P_2(\mu) + \xi^4 P_4(\mu) \right\}.$$

$s=4$:

$$\Theta'=64.4^2 . A_8 \left\{ -\left(144-24\xi^2+\frac{6}{5}\xi^4-\frac{1}{35}\xi^6\right) P_0(\mu) - \left(24\xi^2-\frac{12}{7}\xi^4+\frac{1}{21}\xi^6\right) P_2(\mu) \right.$$

$$\left. -\left(\frac{18}{35}\xi^4-\frac{9}{385}\xi^6\right) P_4(\mu) - \frac{1}{231}\xi^6 P_6(\mu) \right\}.$$

6. We have found out in § 3 that for an angular velocity given by the expression

$$\omega^2=\sum_{s} C_{s+2}\, r^s (1-\mu^2)^{\frac{s}{2}} \quad \ldots \quad (1)$$

the mass M_ω of the rotating star can be put in the form

$$4\pi\lambda \left\{ \frac{k\lambda^{\frac{1}{n}-1}(n+1)}{4\pi\gamma} \right\}^{\frac{3}{2}} \xi_1^2 |\phi'_0(\xi_1)| \left\{ 1+\frac{1}{\xi_1 |\Theta'_1(\xi_1)|} \sum_{s} B_{os} \frac{C_s}{s} I'_s \right\};\; n=1.$$

where B_{os}, I'_s etc., are given in the same section.

For a non-rotating star with the same central density λ and satisfying the polytropic relationship with the same index unity, the mass M_0 is

$$= 4\pi\lambda\left\{\frac{k\lambda^{\frac{1}{n}-1}(n+1)}{4\pi\gamma}\right\}^{\frac{3}{2}}\xi_1^2 \mid \phi'_0(\xi_1) \mid .$$

Hence the mass of the rotating star is greater than that of the non-rotating model in the ratio

$$M_\omega : M_0 = 1 + \frac{1}{\xi_1|\Theta'_1(\xi_1)|}\sum_s B_{os}\frac{C_s}{s}I'_s. \qquad \dots \quad (2)$$

The volume ratio in the same way from the related equations is

$$V_\omega : V_0 = 1 + \frac{3}{\xi_1|\Theta'_1(\xi_1)|}\sum_s B_{os}\frac{C_s}{s}I_s \qquad \dots \quad (3)$$

The ratio of the mean densities for the rotating and non-rotating models can be written down from eq. (10), § 3. These three equations show that the increase in mass, volume or the mean density relative to their values for the non-rotating model is additive when we modify the angular velocity by adding terms of the form (1) in ω^2. The actual values of the ratios can be compared if we wish, for even values of *s*, with the values obtained from usual treatment using eq. (8), § 5.

We shall now write down the relevant values for $s=2$ and $s=4$ from § 3.

$s=2$:

$$\omega^2 = C_2, \text{ the case of uniform rotation ; } \xi_1|\Theta'_1(\xi_1)| = 1$$

$$B_{02} = \frac{4}{4\pi\gamma\lambda};\ I'_2 = -\frac{1}{\pi}\left(\pi - \frac{2\pi^3}{6}\right) = -\left(1 - \frac{\pi^2}{3}\right);\ I_2 = 1$$

$$\therefore \quad M_\omega : M_0 = 1 + \frac{C_2}{2\pi\gamma\lambda}\left(\frac{\pi^2}{3} - 1\right);\ V_\omega : V_0 = 1 + \frac{3C_2}{2\pi\gamma\lambda}$$

$$(\rho_m)_\omega : (\rho_m)_0 = 1 + \frac{C_2}{2\pi\gamma\lambda}\left(\frac{\pi^2}{3} - 4\right).$$

We can compare these with the values given by Chandrasekhar (1933, p. 399).

$s=4$: $\qquad \omega^2 = C_4 r^2(1-\mu^2), \qquad \xi_1|\Theta'_1(\xi_1)| = 1$

$$B_{04} = \frac{R}{4\pi\gamma\lambda}\frac{32/3}{4\pi\gamma\lambda};\ I'_4 = 6\left(1 - \frac{\pi^2}{3} + \frac{\pi^4}{30}\right);\ I_4 = -6\left(1 - \frac{\pi^2}{6}\right)$$

$$\therefore\ M_\omega : M_0 = 1 + \frac{C_4}{2\pi\gamma\lambda}\frac{R}{4\pi\gamma\lambda}.8\left(1 - \frac{\pi^2}{3} + \frac{\pi^4}{30}\right);\ V_\omega : V_0 = 1 + \frac{C_4}{2\pi\gamma\lambda}\frac{R}{4\pi\gamma\lambda}24\left(\frac{\pi^2}{6} - 1\right);$$

$$(\rho_m)_\omega : (\rho_m)_0 = 1 + \frac{C_4}{2\pi\gamma\lambda}\frac{R}{4\pi\gamma\lambda} . 8\left(4 - \frac{5\pi^2}{6} + \frac{\pi^4}{30}\right)$$

For the law taken by Bhatnagar, *viz.*,

$$\omega^2 = \omega_0^2\left\{1 - a\frac{r^2}{R_s^2}(1-\mu^2)\right\};\ R_s^2 = \pi^2\frac{R}{4\pi\gamma\lambda}$$

we get, combining the above,

$$\mathrm{M}_\omega : \mathrm{M}_0 = 1 + \frac{\omega_0^2}{2\pi\gamma\lambda}\left\{\left(\frac{\pi^2}{3} - 1\right) - \frac{8}{\pi^2}\left(1 - \frac{\pi^2}{3} + \frac{\pi^4}{30}\right)\alpha\right\}.$$

$$\mathrm{V}_\omega : \mathrm{V}_0 = 1 + \frac{\omega_0^2}{2\pi\gamma\lambda}\left\{3 - \frac{24}{\pi^2}\left(\frac{\pi^2}{6} - 1\right)\alpha\right\}$$

$$(\rho_m)_\omega : (\rho_m)_0 = 1 + \frac{\omega_0^2}{2\pi\gamma\lambda}\left\{\left(\frac{\pi^2}{3} - 4\right) - \frac{8}{\pi^2}\left(4 - \frac{5\pi^2}{6} + \frac{\pi^4}{30}\right)\alpha\right\}.$$

These can be compared to the values given by Bhatnagar (1940, pp. 34-35).

My grateful thanks are due to Prof. A. C. Banerji for his helpful keen interest in the paper.

MATHEMATICS DEPARTMENT,
ALLAHABAD UNIVERSITY.

References

Bhatnagar, P. L., 1940, *Bull. Cal. Math. Soc.*, 32, 21-41.
Chandrasekhar, S., 1933, *M.N.R.A.S.*, 93, 390-405.
Kopal, Z., 1939, *M.N.R.A.S.*, 99, 266-271.
Milne, E. A., 1923, *M.N.R.A.S.*, 83, 134.
Poincare, Lecons sur les Hypothesis Cosmogoniques, 28.
Zeipel, H. von, 1924, *M.N.R.A.S.*, 84, 691.

ON A MEASURE OF DIVERGENCE BETWEEN TWO STATISTICAL POPULATIONS DEFINED BY THEIR PROBABILITY DISTRIBUTIONS

By

A. Bhattacharyya

(Communicated by Mr. R. C. Bose—Received June 2, 1948)

Although the theories of Estimation and of Testing of Hypothesis are fairly developed, that of the Measure of Divergence is still in the rudimentary state. Except in the case of two normal populations (univariate or multivariate) differing in their means but with the same system of variances and covariances, and in the univariate case differing in their variances, no measure of any practical use has been defined. The author (Bhattacharyya, 1941) defined the divergence between two multinomial populations and from this, by passing to the limit, got a measure of divergence between two populations defined in any way and having the same number of variates in which an (1, 1) correspondence can be established. This was applied in the above cases already mentioned and the results were found to agree with the accepted measures. Now we want to give some reasons in favour of our adopted measure of divergence, though the results were obtained early in 1941.

2. Consider two multinomial populations characterised by the population probabilities $(\pi_1, \pi_2, \ldots \pi_k)$ and $(\pi_1', \pi_2', \ldots \pi_k')$ of the k exhaustive and mutually exclusive ways of occurrence. Then as $\sum_{i=1}^{k} \pi_i = 1$ and $\sum_{i=1}^{k} \pi_i' = 1$, $(\sqrt{\pi_1}, \sqrt{\pi_2} \ldots\ldots \sqrt{\pi_k})$ and $(\sqrt{\pi_1'}, \sqrt{\pi_2'}, \ldots\ldots \sqrt{\pi_k'})$ can be considered as the direction cosines of two straight lines in k-dimensional space referred to a system of orthogonal co-ordinate axes. These lines may be called population lines. As a measure of divergence between the two populations, we have adopted the square of the angle between these two lines. The angle itself may be called the angle of divergence. But it is obvious that any monotonically increasing single-valued function of this angle may be used as a measure of divergence. Then, if Δ is the angle of divergence, we have

$$\operatorname{Cos} \Delta = \sum_{i=1}^{k} \sqrt{\pi_i \, \pi_i'} \qquad \ldots \quad (2.1)$$

Some reasons for using the above geometrical representation become clear if we consider the sample. Let a sample of size n be taken from the first population which can be well considered to be an urn containing balls of k different colours where the sampling is with replacement. Let $n_1, n_2, \ldots n_k$ be the number of balls of different colours in the sample of size n with relative frequencies $(p_1, p_2, \ldots p_k)$, where $p_i = \dfrac{n_i}{n}$.

(1) If we represent the sample by a point $(\sqrt{n_1}, \sqrt{n_2}, \ldots \sqrt{n_k})$ in k-dimensions, then all points which represent samples of size n lie on a hypersphere of

radius $\sqrt{n}$ about the origin. If we change the size keeping the relative frequencies fixed the new points lie on a line through the origin. Thus, the change in the size is denoted by a radial change, but the change in the relative proportion by a movement on the same sphere. These movements are orthogonal and so can be considered to be independent. So, the proportions have been disentangled from the size. The direction cosines of the sample line joining the sample point to the origin are $(\sqrt{p_1}, \sqrt{p_2}, \cdots \sqrt{p_k})$. When the sample size is increased indefinitely, the proportions (p_i) coincides with the population probabilities (π_i) and we get the population line. (2) Another reason is that the standard deviation of each n_i is proportional to $\sqrt{n}$ and so, that of $\sqrt{n_i}$ is independent of n to the first approximation. So, $\sqrt{n_i}$'s are more suitable for comparison. (3) A third reason is interesting. The divergence between the sample and the population is easily got from the relation

$$\operatorname{Cos} D' = \sum_{i=1}^{k} \sqrt{p_i \pi_i} \quad \ldots \quad (2.2)$$

where D′ is the angle of divergence between the two. Hence

$$4 \sin^2 \frac{D'}{2} = \sum_{i=1}^{k} (\sqrt{p_i} - \sqrt{\pi_i})^2 \quad \ldots \quad (2.3)$$

$$= \sum_{i=1}^{k} \frac{(p_i - \pi_i)^2}{(\sqrt{p_i} + \sqrt{\pi_i})^2} = \sum_{i=1}^{k} \frac{(p_i - \pi_i)^2}{4\pi_i}, \text{ approximately}$$

correct up to terms of order $\left(\frac{1}{n}\right)$,

$$= \frac{\chi^2}{4n}$$

where χ^2 is the Pearsonian coefficient for goodness of fit. And as χ^2 has the mean and variance of the order of k, $\frac{\chi^2}{n}$ is small when n is large and so, $\sin \frac{D'}{2} = \frac{D'}{2}$. Hence we get, approximately

$$D' = \frac{\chi}{2\sqrt{n}}. \quad \ldots \quad (2.4)$$

Similarly we can show that to the same order of approximation

$$4 \sin^2 \frac{D'}{2} = \sum_{i=1}^{k} \frac{(p_i - \pi_i)^2}{2p_i}$$

and hence

$$D' = \frac{\chi'}{2\sqrt{n}} \quad \ldots \quad (2.5)$$

where χ'^2 has been proposed by Jeffreys and others for testing the goodness of fit alternatively to χ^2 (Jeffreys, 1938). Thus we see that to the same order of approximation as χ or χ' we can use $\sqrt{n}$ D′ for the same purpose. Moreover this is symmetric with respect to p_i and π_i.

8. Let us now put forward some argument for our proposed coefficient of divergence. As before an urn containing balls of k different kinds where the sampling is with replacement can be a physical picture of a multinomial population. For simplicity consider an urn containing balls of three different colours, red, white and black, with proportions π_1, π_2 and π_3. An urn containing balls of only one colour, say red, has got the probabilities 1, 0, 0. In this second urn red ball is a certainty. The angle of divergence Δ_1 between these two multinomial populations is, from definition, given by

$$\cos \Delta_1 = \sqrt{\pi_1}. \qquad \ldots \quad (8.1)$$

Thus, π_1, the probability of occurrence of the red ball, is the square of the cosine of the angle of divergence between the 1st population with the second in which red ball is a certainty. Similarly, for the other probabilities we can get interpretations in terms of divergences, the result being true in the general multinomial population. In fact squares of the cosines of the angles of divergence can be considered as generalised probability. Consider now that the second urn instead of containing only balls of red colour, also contains balls of blue colour, with proportions π_1' and π_4'. Now, the divergence between these two populations represented by the urns is dependent on only the proportion of balls having the same colour, which alone determines the resemblance between the two populations, and must be independent of the proportions of balls of different colours. If Δ_{12} be the angle of divergence between these two urns, then

$$\cos \Delta_{12} = \sqrt{\pi_1 \pi_1'} + \sqrt{\pi_2 . 0} + \sqrt{\pi_3 . 0} + \sqrt{0 . \pi_4'} = \sqrt{\pi_1 \pi_1'}$$

Now if we consider a new multinomial population, the elements of which consist of the drawing of two balls, one from each urn then the probability of getting a double red is $\pi_1 . \pi'_1$. On the other hand had the two urns contained entirely red balls, double drawing of red balls would have been a certainty and the angle of divergence Δ between these two populations so constructed would have been the same as given just now, namely $\cos \Delta = \sqrt{\pi_1 \pi'_1}$. Thus $\Delta_{12} = \Delta$. Again, if the two urns contain balls of entirely different kinds, say the first one black and white and the second one red and blue, then there is no resemblance between the two urns at all. They are entirely different. In this case the angle of divergence attains the maximum value of 90°. And this holds so long as there is not any ball of the same colour in both the urns, and conversely. On the other hand, two urns are identical so far as sampling with replacement is concerned if the proportions of balls of different colours are the same in the two urns, and conversely. Thus if the 1st and the 2nd urn both have red and blue balls with the same proportions π_1 and π_2, then the angle of divergence Δ is zero.

The physical considerations inherent in the notion of divergence between two multinomial populations can be summarised from the above considerations as follows :—

(*i*) Two multinomial populations can be considered to be identical if the probabilities characterising the two be the same and conversely ;

(*ii*) Two multinomial populations may be considered to be entirely different if there are no ways of occurrence common to both, *i.e.*, if they are mutually exclusive and conversely;

(*iii*) The resemblance between two multinomial populations, if that can be conceived to exist at all, must depend on the ways of occurrence common to the two populations and must be independent on the ways of occurrence which are not common to these;

(*iv*) The resemblance between a multinomial population and another in which one of the ways of occurrence is a certainty must depend on the probability of this particular way of occurrence in the 1st population. Our adopted measure of divergence satisfies all these considerations as can be easily seen. For the definition (2.1) of the angle of divergence shows that if $\Delta=90°$, then $\sum_{i=1}^{k}\sqrt{\pi_i\pi'_i}=0$ and as each term is positive, $\pi_i\pi'_i=0$ for all values of i, which satisfies (*ii*). Again (2.3) shows that if $\Delta=0$, then $\sum_{i=1}^{k}(\sqrt{\pi_i}-\sqrt{\pi'_i})^2=0$ and consequently $\pi_i=\pi'_i$ for all values of i. So, (*i*) is satisfied. Similarly, from (2.1) it is seen that (*iii*) and (*iv*) are both satisfied.

4. Here we consider the possibility of using a monotonically increasing single valued function of Δ as a measure of divergence. In the simple case of Bernoullian distribution where there are two possibilities, success and failure, let there be two populations with probabilities π_1, π_2 and π'_1, π'_2. The angle of divergence as determined from these populations is given by

$$\cos\Delta=\sqrt{\pi_1\pi'_1}+\sqrt{\pi_2\pi'_2}. \qquad \dots \quad (4.1)$$

But if we take two samples of size n one from each population, then the distribution of k successes is given by the theorem of Bernoulli as ${}^nC_k\pi_1^k\pi_2^{n-k}$ and ${}^nC_k\pi'^k_1\pi'^{n-k}_2$ and the angle of divergence Δ_n between these two samples is obtained from

$$\cos\Delta_n=\sum_k\sqrt{\{{}^nC_k\pi_1^k\pi_2^{n-k}\}\{{}^nC_k\pi'^k_1\pi'^{n-k}_2\}}$$
$$=(\sqrt{\pi_1\pi'_1}+\sqrt{\pi_2\pi'_2})^n=(\cos\Delta)^n. \qquad \dots \quad (4.2)$$

And so, as the actual angle divergence is Δ.

$$\cos\Delta=(\cos\Delta_n)^{1/n}=1+\frac{\log\cos\Delta_n}{n}+\frac{(\log\cos\Delta_n)^2}{2n^2}+\dots$$

i e.,

$$-2\sin^2\frac{\Delta}{2}=\frac{\log\cos\Delta_n}{n}+\dots$$

So, when n is very large and the two populations are not widely different (*i.e.*, $\log\cos\Delta_n$ is not $O(n)$) we easily see that

$$\Delta^2=-2\log\cos\Delta_n/n. \qquad \dots \quad (4.3)$$

Thus $-\log\cos\Delta_n$ can be used as a measure of divergence between two binomial populations when this is estimated from two sufficiently numerous repeated trials, In the case of two Poisson's distributions whose means are m and m' the angle of divergence is given by

$$\cos\Delta' = \sum_{x=0}^{\infty}\left[e^{-m}.\frac{m^x}{x!}\right]^{\frac{1}{2}}\left[e^{-m'}\frac{m'^x}{x!}\right]^{\frac{1}{2}}$$

$$= e^{-\frac{m}{2}-\frac{m'}{2}+\sqrt{mm'}} = e^{-\frac{1}{2}(\sqrt{m}-\sqrt{m'})^2} \qquad \ldots \quad (4.4)$$

As there is only one parameter in this distribution $(\sqrt{m}-\sqrt{m'})^2$ can be used as a measure of divergence and hence $-\log\cos\Delta'$ can well be used for the same purpose in this case.

Another interesting result is that if we have two pairs of binomial populations with probabilities (π_1, π_2), (π_3, π_4) and (π'_1, π'_2), (π'_3, π'_4) then the joint probabilities of these two pairs are $(\pi_1\pi_3, \pi_1\pi_4, \pi_2\pi_3, \pi_2\pi_4)$ and $(\pi'_1\pi'_3, \pi'_1\pi'_4, \pi'_2\pi'_3, \pi'_2\pi'_4)$ and so if $\Delta_{12,\,34}$, Δ_{13}, Δ_{24} be the angles of divergence between the 1st pair and the 2nd pair, the two firsts and the two lasts of the two pairs respectively, then

$$\cos\Delta_{12,\,34} = (\sqrt{\pi_1\pi'_1}+\sqrt{\pi_2\pi'_2})(\sqrt{\pi_3\pi'_3}+\sqrt{\pi_4\pi'_4}) = \cos\Delta_{13}.\cos\Delta_{24}$$

which can be considered to be a generalisation of the multiplication theorem of probability. But if we define divergence to be $-\log\cos\Delta$ then we get

$$-\log\cos\Delta_{12,\,34} = -\log\cos\Delta_{13} - \log\cos\Delta_{24} \qquad \ldots \quad (4.5)$$

where we see that by adding two divergences we deduce that of their compound.

In the case of two multinomial populations exactly similar results hold as given above in (4.3) and (4.5).

5. So far we have kept ourselves confined to the region of discrete distributions but now we consider continuous distributions. There is no logical difficulty in defining divergence in these cases, for we can directly pass to the limit and so from (2.1) we get, if Δ is the angle of divergence,

$$\cos\Delta = \int\sqrt{\phi(x_1, x_2, \ldots x_k).\phi'(x_1, x_2, \ldots x_k)}\; dx_1dx_2 \ldots dx_k \qquad \ldots \quad (5.1)$$

where the integration is performed over the whole space and where the probability distributions of variates x_1, x_2, ... x_k in the two cases are

$$\left.\begin{array}{l}\phi\,(x_1, x_2, \ldots x_k)\; dx_1\; dx_2 \ldots dx_k \\[1em] \phi'(x_1, x_2, \ldots x_k)\; dx_1\; dx_2 \ldots dx_k\end{array}\right\} \qquad \ldots \quad (5.2)$$

respectively. Here the only assumptions are made that we can get an (1, 1) correspondence between the variates in the two cases.

Again as in section 4, if $\phi(x_1, x_2, \ldots x_k)$, $\psi(y_1, y_2, \ldots y_l)$ and $\phi'(x_1, x_2 \ldots x_k)$, $\psi'(y_1, y_2, \ldots y_l)$ be two pairs of probability distributions, and if $\Delta_{12,\,34}$ be the angle of divergence of the 1st pair from the 2nd pair, then

$$\cos\Delta_{12,\,34} = \iint\sqrt{\phi\phi'\psi\psi'}\; dx_1dx_2 \ldots dx_k\; dy_1dy_2 \ldots dy_l$$

$$= \int\sqrt{\phi\phi'}\; dx_1dx_2 \ldots dx_k \times \int\sqrt{\psi\psi'}\; dy_1dy_2 \ldots dy_l$$

a result similar to (4.5).

6. Let us apply the above method to find the divergence between two univariate normal populations differing in their means and standard deviations. The angle of divergence is given by

$$\cos\Delta = \frac{1}{\sqrt{2\pi\sigma_1\sigma_2}}\int_{-\infty}^{\infty} e^{-\frac{(x-m_1)^2}{4\sigma_1^2}-\frac{(x-m_2)^2}{4\sigma_2^2}}\,dx$$

as defined in (5.1). Integrating, it is easily seen that

$$\cos\Delta = \sqrt{\frac{2\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2}}\cdot e^{-\frac{(m_1-m_2)^2}{4(\sigma_1^2+\sigma_2^2)}} \qquad \ldots \quad (6.1)$$

(*i*) If now $\sigma_1=\sigma_2=\sigma$, then

$$\cos\Delta = e^{-\frac{(m_1-m_2)^2}{8\sigma^2}}$$

or,

$$\frac{(m_1-m_2)^2}{\sigma^2} = -8\log\cos\Delta. \qquad \ldots \quad (6.2)$$

Now the expression in the left side is the universally accepted measure of divergence in the case of two normal populations differing in their means only. As remarked, in section 4, $-\log\cos\Delta$ is also a suitable measure of divergence, being a monotonically increasing single-valued function of Δ whose range of variation is from 0 to ∞, whereas that of Δ is only from 0 to 90°. So here we get consistent result by our method. The reason for this function of Δ being suitable as a measure of divergence is that the binomial distribution tends to normality in large samples. If we want to estimate this measure of divergence in terms of sample readings, then we get the estimate, by putting efficient statistics in place of the parameters, as

$$(\bar{x}_1-\bar{x}_2)^2 \Big/ \frac{n_1s_1^2+n_2s_2^2}{n_1+n_2-2}.$$

The connection of this to the well known Fisher's '*t*' is obvious. This being the sample estimate of the population divergence, is the most suitable statistic to test the null hypothesis regarding the equality of the means.

(*ii*) If $m_1=m_2$, then

$$\cos\Delta = \sqrt{\frac{2\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2}}. \qquad \ldots \quad (6.3)$$

Here of course we may choose any simple monotonically increasing function of Δ as a measure of divergence. Now $\frac{\sigma_1^2}{\sigma_2^2}$ is such a function. This can be chosen. Similarly $\frac{1}{2}\log(\sigma_1^2/\sigma_2^2)$ is another such function. Both these functions are asymmetrical and so suffers from this disadvantage. But their simplicity compensates more than what is lost in symmetry. The sample estimates are $\frac{s_1^2}{s_2^2}$ and $\frac{1}{2}\log(s_1^2/s_2^2)$ which are the well known statistics of Fisher and is used to test the null hypothesis.

(*iii*) If the ratio of σ_1/σ_2 is fixed, then

$$\cos\Delta=\left\{\tfrac{1}{2}\left(\frac{\sigma_1}{\sigma_2}+\frac{\sigma_2}{\sigma_1}\right)\right\}^{-\frac{1}{2}} e^{-\frac{(m_1-m_2)^2}{4(\sigma_1^2+\sigma_2^2)}}$$

But the first factor being fixed we can take the measure of divergence as

$$(m_1-m_2)^2/(\sigma_1^2+\sigma_2^2).$$

Substituting the efficient estimates of the parameter we get the sample estimate as $(\bar{x}_1-\bar{x}_2)^2/(s_1^2+s_2^2)$, the root of which has been proposed to test the equality of the two means when nothing is known regarding the S.D.'s (Fisher, 1985). By our method we see that σ_1/σ_2 aught to be known in order that the above can be used, which in its probability distribution involves this ratio. But Fisher, by means of Fiducial argument circumvented this restriction.

7. We shall now consider some of the aspect of transformation of variates to make the variance constant in connexion with the notion of divergence.

(*i*) In the case of the Binomial distribution in n repeated trials we know that the sample proportion is distributed approximately normally about the population proportion π with variance $\pi(1-\pi)/n$. But the variance depends on π. It can be shown that $\sin^{-1}\sqrt{p}$ is distributed about $\sin^{-1}\sqrt{\pi}$ with approximately constant variance $1/4n$. Hence in two Binomial populations with population probabilities π and π' and having the same size n, the angle of divergence is given by (from 6.1)

$$-(2\log\cos\Delta)/n=(\sin^{-1}\sqrt{\pi}-\sin^{-1}\sqrt{\pi'})^2.$$

The right side is easily seen to be

$$[\cos^{-1}\{\sqrt{\pi\pi'}+\sqrt{(1-\pi)(1-\pi'}\}]^2$$

which is the population divergence as defined in (2.1). The above result is the same as obtained in (4.8). Similarly in the case of Poisson's distribution, if we assume that $\sqrt{x}$ is distributed about $\sqrt{m}$ (where m is the mean) normally with a constant variance, we get the result already obtained in (4.4).

(*ii*) For the case of the standard deviations of the samples of sizes n_1 and n_2 from two populations having S.D.'s σ_1 and σ_2, the angle of divergence is given by

$$\cos\Delta=\frac{\text{const.}}{\sigma_1^{n_1-1}\sigma_2^{n_2-1}}\int_0^\infty\left(e^{-\frac{n_1s^2}{\sigma_1^2}}s^{n_1-2}\cdot e^{-\frac{n_2s^2}{\sigma_1^2}}s^{n_2-2}\right)^{\frac{1}{2}}ds$$

$$=\text{const.}\left(\frac{n_1}{\sigma_1^2}+\frac{n_2}{\sigma_2^2}\right)^{-\frac{n_1+n_2-2}{2}}\Big/\left(\sigma_1^{n_1-1}\ \sigma_2^{n_2-1}\right)$$

$$=\text{const.}\left(\frac{\sigma_1}{\sigma_2}\right)^{n_2-1}\left(n_1+n_2\frac{\sigma_1^2}{\sigma_2^2}\right)^{-\frac{n_1+n_2-2}{2}}$$

From this it is seen that Δ is a single-valued increasing function of σ_1/σ_2. So either σ_1/σ_2 or $\log(\sigma_1/\sigma_2)$ may be taken as a measure of divergence, although both of these values lack symmetry. If we however use the result that $\log s$ is distributed about $\log \sigma$ normally with variance approximately equal to $1/2n$ then we get (from 6.1).

$$\cos \Delta = \text{const. } e^{-[\log (\sigma_1/\sigma_2)]^2/2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$

and so $$[\log (\sigma_1/\sigma_2)]^2 = \text{const.} - 2(\log \cos \Delta)\left(\frac{1}{n_1}+\frac{1}{n_2}\right).$$

This result is in conformity with the above. In fact $[\log (\sigma_1/\sigma_2)]^2$ is a symmetrical expression in σ_1 and σ_2 which is an additional advantage.

In these cases of those transformations which make the variance approximately independent of the parameters, the divergence calculated from the transformed variates with the assumption of normality gives results not in contradiction with accepted notions.

8. Let us now investigate the case of two multivariate normal populations differing in their means, but having the same set of variances and covariances. Let the two sets of means be $(m_1, m_2, \ldots m_k)$ and $(m'_1, m'_2, \ldots m'_k)$ and the same matrix of variances and covariances $((a_{ij}))$. Let A be the determinant $((a_{ij}))$. Then from (5.1), we get

$$\cos \Delta = \frac{1}{(2\pi)^{\frac{k}{2}}\sqrt{A}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{4}\sum\limits_{i,j} a^{ij}\{(x_i-m_i)(x_j-m_j)+(x_i-m'_i)(x_j-m'_j)\}} \prod_i dx_i \qquad \ldots \quad (8.1)$$

where $((a^{ij}))$ is the reciprocal matrix of $((a_{ij}))$. Let

$$\sum_{i,j} a^{ij}\{(x_i-m_i)(x_j-m_j)+(x_i-m'_i)(x_j-m'_j)\} = 2\sum_{i,j} a^{ij}(x_i-m''_i)(x_j-m''_j)+C \qquad \ldots \quad (8.2)$$

identically where m''_i, $i=1, 2, \ldots k$ and C are yet undetermined. This way of representing the left side is always possible. Assuming (8.2), we get

$$\cos\Delta = \frac{1}{(2\pi)^{\frac{k}{2}}\sqrt{A}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum a^{ij}(x_i-m''_i)(x_j-m''_j)-\frac{1}{4}C} \prod dx_i$$

$$= e^{-\frac{1}{4}C} \qquad \ldots \quad (8.3)$$

To evaluate C, we note that (8.2) holds identically. So differentiating both sides with respect to x_j we get

$$\sum_{i=1}^{k} a^{ij}(x_i-m_i+x_i-m'_i) = 2\sum_{i=1}^{k} a^{ij}(x_i-m''_i), \quad j=1, 2, \ldots k,$$

that is $$\sum_{i=1}^{k} a^{ij}(2m''_i-m_i-m'_i)=0, \quad j=1, \ldots k.$$

So, multiplying each of these k equations by $2m''_j-m_j-m'_j$, $j=1, 2, \ldots k$ respectively and adding, we get

$$\sum_{i,j} a^{ij}(2m''_i-m_i-m'_i)(2m''_j-m_j-m'_j)=0. \qquad \ldots \quad (8.4)$$

But if in the identity (8.2) we put $x_i = \frac{1}{2}(m_i + m'_i)$ we get

$$\tfrac{1}{2} \sum_{i,j} a^{ij}(m_i - m'_i)(m_j - m'_j) = 2 \sum_{i,j} a^{ij}\left(\frac{m_i + m'_i}{2} - m''_i\right)\left(\frac{m_j + m'_j}{2} - m''_j\right) + C.$$

Hence

$$C = \tfrac{1}{2} \sum_{i,j} a^{ij}(m_i - m'_i)(m_j - m'_j), \qquad \dots \quad (8.4)$$

for the first term on the left side vanishes by virtue of the relation (8.3). From these it is seen that

$$-8 \log \cos \Delta = \sum_{i,j} a^{ij}(m_i - m'_i)(m_j - m'_j)$$

The right side of the above, except for a factor, has been used as a measure of divergence of two multinomial populations by Bose and Roy under the same hypothesis as ours (Bose and Roy, 1938).

9. As in the case of multinomial populations where each population was represented as a line in k-dimensions or as a point on the unit hyper-sphere, we can consider a continuous population as a line in infinite dimensions or as a point on a unit hypersphere (where the line meets it) in the same space. The divergence is denoted by the angle between the lines denoting the populations or by the arcual distance between the two points on the unit hypersphere representing the two populations. Consider a normal population with mean m and S.D. σ. By varying m and σ we get different points on the unit hypersphere all of which lie in a two dimensional band. Among these points (m, σ) the point $(0, 1)$ can be considered to be the starting point, the two systems of lines $m = \text{const.}$ and $\sigma = \text{const.}$ can be regarded as the two curvilinear co-ordinate system in the two dimensional band. The lines of reference can be taken to be the lines $m = 0$ and $\sigma = 1$. The length of the curve from $\sigma = 1$ to σ on the first system of lines $m = \text{const.}$ can be obtained as follows: if δs be the length of the arc between two consecutive points σ and $\sigma + \delta\sigma$ (where δ denotes small increment), then

$$\cos \delta s = \sqrt{\frac{2\sigma(\sigma + \delta\sigma)}{\sigma^2 + (\sigma + \delta\sigma)^2}}, \quad \therefore \quad \delta s = \frac{\delta\sigma}{\sqrt{2}\sigma}, \quad \therefore \quad s = \frac{1}{\sqrt{2}} \log \sigma$$

Similarly, the length of the curve from $m = 0$ to m on the line $\sigma = \text{const.}$ can also be got as follows: if $\delta s'$ be the length of the arc between two consecutive points m and $m + \delta m$, then

$$\cos \delta s' = e^{-\frac{(\delta m)^2}{8\sigma^2}}, \quad \therefore \quad \delta s' = \frac{\delta m}{2\sigma}, \quad \therefore \quad s' = \frac{m}{2\sigma}.$$

So we get geometrical representation for m and σ in a normal population.

An interesting interpretation of intrinsic accuracy is obtained when we apply the above analysis to uniparametric distributions. Let $f(x, \theta)dx$ be the probability distribution of x. Then as before the distribution, for a given value of θ, is represented by a point in the unit hypersphere and by changing θ we get a curve on it. Let s be the distance along the curve from any fixed point to the point θ, then as before, the arcual distance δs between two points θ and $\theta+\delta\theta$ is given by

$$\cos \delta s=\int_{-\infty}^{\infty} \sqrt{f(x, \theta).f(x, \theta+\delta\theta)}\,dx.$$

Now, by Taylor's theorem

$$\sqrt{f(x,\theta+\delta\theta)}=f^{\frac{1}{2}}+\delta\theta\frac{f'}{2f^{\frac{1}{2}}}+\frac{(\delta\theta)^2}{2!}\left\{\frac{f''}{2f^{\frac{1}{2}}}-\frac{(f')^2}{4f^{\frac{3}{2}}}\right\}+\ldots$$

$$\therefore \quad \cos\delta s=\int_{-\infty}^{\infty}\left\{f+\delta\theta\frac{f'}{2}+\frac{(\delta\theta)^2}{2}\left(\frac{f''}{2}-\frac{f'^2}{4f}\right)+\ldots\right\}dx$$

$$=1-\frac{(\delta\theta)^2}{8}\int_{-\infty}^{\infty}\frac{f'^2}{f}dx+\ldots$$

∴ proceeding to the limit, we get

$$\left(\frac{ds}{d\theta}\right)^2=\frac{1}{4}\int_{-\infty}^{\infty}\frac{f'^2}{f}dx=-\frac{1}{4}\int_{-\infty}^{\infty}f\ \frac{\partial^2}{\partial\theta^2}\log f\,dx=\tfrac{1}{4}\mathrm{I}, \quad \ldots \quad (9.1)$$

where I is the intrinsic accuracy of the frequency distribution of x as defined by Fisher.

If we have two distributions y_1dx and y_2dx both involving the parameter θ, then in order to compare the two distributions we may consider the relative rate of change of the elementary arcual distances as the parameter θ is changed. Now the relative rate of change of arcs along the two curves defined by the two distributions is

$$\frac{ds_1}{ds_2}=\left(\frac{ds_1}{d\theta}\right)\left(\frac{d\theta}{ds_2}\right)=\sqrt{\frac{\mathrm{I}_1}{\mathrm{I}_2}} \quad \ldots \quad (9.2)$$

which is the square root of relative efficiency of the two distributions.

10. We have so far considered the divergence of two populations, discrete or continuous. But we can conceive of cluster divergence among a number of populations. Though here we cannot get any simple geometrical representation as has been obtained in the above cases, still we can, by analogy, get a measure which vanishes when all the populations are identical and attains its maximum value when there is nothing common to all of them. For instance, as before, if there are more than two urns having proportions of balls of different colours the same, then the divergence is zero and if there is not a single ball of the same colour in all of them, then the divergence is maximum. Let there be n multinomial populations $(\pi_1^i, \pi_2^i, \ldots \pi_k^i)$, $i=1, 2, \ldots n$. Then if

$$\vartheta=\sum_{j=1}^{k}\left(\prod_{i=1}^{n}\pi_j^i\right)^{\frac{1}{n}}, \quad \ldots \quad (10.1)$$

$-\log\vartheta$ or any monotone increasing single-valued function of this may be used as a measure of divergence. If all the multinomial populations have got the same set of probabilities, then $\log\vartheta=0$. This is both necessary and sufficient. Also, if $\vartheta=0$, then as the terms on the right side of (10.1) are all non-negative, each of the terms must vanish. So, there cannot be any common event in all the populations. Also, as $\vartheta \leq 1$, $-\log \vartheta$ is positive. So this expression satisfy the above-mentioned requirements. It can be easily shown that results similar to (4.3) and (4.5) hold here also. In the case of continuous populations we get

$$\vartheta = \int \ldots \int \sqrt[n]{\phi_1 \cdot \phi_2 \ldots \phi_n}\; dx_1 dx_2 \ldots dx_k. \qquad \ldots \quad (10.2)$$

If we apply this to the univariate normal populations differing in their means only we get

$$\vartheta = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\Sigma(x-m_i)^2}{2n\sigma^2}} dx = e^{-\frac{\Sigma m_i^2}{2n\sigma^2} + \frac{(\Sigma m_i)^2}{2n^2\sigma^2}}$$

$$\therefore \quad -2n \log \vartheta = [\Sigma m_i^2 - n(\bar{m})^2]/\sigma^2,$$

where $$\bar{m} = (\Sigma m_i)/n.$$

This result resembles the familiar analysis of variance, the numerator being the variance of the means and the denominator, the error variance.

In the case where the means are all equal, but the standard deviations are different, we get

$$\vartheta = \frac{1}{\sqrt{2\pi}(\sigma_1 \sigma_2 \ldots \sigma_n)^{\frac{1}{n}}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2n} \cdot \Sigma \frac{1}{\sigma_i^2}} dx$$

$$= \sqrt{\left[\frac{\left(\frac{1}{\sigma_1^2} \cdot \frac{1}{\sigma_2^2} \cdots \frac{1}{\sigma_n^2} \right)^{\frac{1}{n}}}{\frac{1}{n}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \ldots + \frac{1}{\sigma_n^2} \right)} \right]}.$$

So, here the right hand side is the square root of the ratio of the geometric mean of the invariances or the amounts of information to their arithmetic mean.

The expression for cluster divergence gives us an expression for the divergence between two populations with different weights. Thus, if the n multinomial populations in (10.1) be resolved into two groups of n_1 and n_2 populations such that the cluster divergence within the two groups are separately zero, then (10.1) reduces to

$$\vartheta = \sum_{i=1}^{k} \left(\pi_i^{n_1} \pi_i'^{n_2} \right)^{\frac{1}{n_1+n_2}}$$

which can be considered to be a generalisation of (2.1). It is hoped to publish further results along this line in future.

References

Bhattacharyya, A., 1942, *Proc. Ind. Sc. Congress.*
Bose, R. C. and Roy, S. N., 1938, *Sankhyā*, 4
Jeffreys, H., 1939, *Theory of Probability*.

ON S-GROUPS

BY

M. A. JABBER

(Received July 3, 1943)

In a recent paper (Levi, 1942) groups were investigated in which the commutator operation satisfies certain algebraic conditions, *e.g.*, in which the commutator operation is associative. This latter class of groups was called S-groups. The methods developed there for the enumeration of S-groups with two generators will be generalised in the present paper for the investigation of S-groups with any finite number n of generators.

§ 1. Let G be an S-group with a finite number of generators $a_1, a_2, \ldots a_n$. Denote the commutator $a_i a_j a_i^{-1} a_j^{-1} = (a_i, a_j)$ by A_{ij}. Since in an S-group all the commutators lie in the centre, every element of G can be written as

$$s = a_1^{p_1} a_2^{p_2} \ldots\ldots a_n^{p_n} A_{12}^{P_{12}} \ldots\ldots A_{n-1\,n}^{P_{n-1,n}}$$

or symbolically,

$$s = [p_1, p_2, \ldots p_n;\; P_{12}, P_{13} \ldots P_{n-1,n}]. \qquad \ldots \quad (1)$$

Let

$$s_1 = [p_1^1, p_2^1, \ldots p_n^1;\; P_{12}^1, P_{13}^1, \ldots P_{n-1,n}^1]$$

$$s_2 = [p_1^2, \ldots p_n^2;\; P_{12}^2, \ldots P_{n-1,n}^2]$$

then direct calculation shows that

$$s_1 s_2 = [q_1, \ldots q_n;\; Q_{12}, \ldots Q_{n-1,n}],$$

where

$$\begin{aligned} q_i &= p_i^1 + p_i^2 \\ Q_{ij} &= P_{ij}^1 + P_{ij}^2 - p_j^1 p_i^2 \end{aligned} \qquad \ldots \quad (2)$$

and that the associative law holds for all the p, q, P and Q's, which are supposed to be integral numbers. If on the other hand, one defines elements by the help of the ordered sets (1) of integral numbers, and determines a 'composition' by (2), then for this composition, the associative law holds. The element $[o, o, \ldots o, o, \ldots o]$ has the properties of the unit element and to every element (1), there exists an inverse element

$$s^{-1} = [-p_1, \ldots -p_n;\; -P_{12} - p_1 p_2, \ldots -P_{n-1,n} - p_{n-1} p_n]. \qquad \ldots \quad (3)$$

Hence these elements form a group. The commutator of two elements s_1 and s_2 is

$$(s_1, s_2) = \left[o, \ldots o; \begin{vmatrix} p_1^1 p_2^1 \\ p_1^2 p_2^2 \end{vmatrix}, \ldots \begin{vmatrix} p_{n-1}^1 p_n^1 \\ p_{n-1}^2 p_n^2 \end{vmatrix} \right]$$

and

$$((s_1, s_2), s_3) = [o, \ldots o; o, \ldots o]$$

So this group is an S-group. For the purpose of this paper, it may be called the *free S-group*, generated by $a_1, a_2, \ldots a_n$. It may be emphasised that two elements of a free S-group are equal if and only if, the corresponding co-ordinates in representation (1) are equal. In general, an S-group generated by $a_1, a_2, \ldots a_n$

satisfies (2) and (3), but the same element might be represented by various systems (1) of co-ordinates. Hence

Every S-group generated by n elements is homomorphic to the free S-group generated by $a_1, a_2, \ldots a_n$.

§ 2. To investigate the S-groups G with n generators, one has to classify the various system of relations which might hold. Denote the commutator group of G by C, then both C and G/C are abelian groups. The relations in G/C are of the form

$$\left.\begin{aligned} a_1^{k_1^1} \ldots a_n^{k_n^1} &\equiv 1 \\ a_1^{k_1^n} \ldots a_n^{k_n^n} &\equiv 1 \end{aligned}\right\} (\text{mod } \mathrm{C}) \qquad \ldots \quad (4)$$

As the a's are commutative modulo C, the vectors $(k_1^i, \ldots k_n^i)$ form a vector-space of n-vectors, hence the maximum number of independent relations in (4) is n.

Applying the theory of elementary divisors to the matrix of the indices of the relations (4), one gets

$$\mathrm{K} = \mathrm{P} \begin{pmatrix} e_1 & & & \\ & e_1 e_2 & & \\ & & \ldots & \\ & & & e_1 e_2 \ldots e_n \end{pmatrix} \mathrm{Q}$$

where $\mathrm{K} = ((k_j^i)),\ \mathrm{P} = ((p_j^i)),\ \mathrm{Q} = ((q_j^i))$

and $\det \mathrm{P} = \pm 1,\ \det \mathrm{Q} = \pm 1.$

Some of the e's may be equal to zero, so that the subsequent e's may be taken arbitrarily and will be selected equal to zero. With this restriction, the e's are invariant under all transformations of the matrix. If in particular, G is a free S-group, all the e's are zeros.

So by the transformation

$$\begin{aligned} a_1 &\longrightarrow a_1^{p_1^1} \ldots a_n^{p_n^1} \\ a_n &\longrightarrow a_1^{p_1^n} \ldots a_n^{p_n^n} \end{aligned}$$

the relations (4) reduce to

$$\left.\begin{aligned} a_1^{e_1} &\equiv 1 \\ a_n^{e_1 \ldots e_n} &\equiv 1 \end{aligned}\right\} (\text{mod } \mathrm{C}) \qquad \ldots \quad (5)$$

The commutator group not being changed by these transformations.

By similar consideration of the relations of the commutator group C, which is also abelian, one gets them in the form

$$b_1^{f_1} = b_2^{f_1 f_2} = \ldots = b_m^{f_1 f_2 \ldots f_m} = 1 \qquad \ldots \quad (6)$$

where

$$\left.\begin{aligned} \mathrm{A}_{i,j} &= b_1^{q_1} b_2^{q_2} \ldots b_m^{q_m} \\ q_l &= p_l^{ij},\ m = \tbinom{n}{2} \\ \det\ &((p_k^{ij})) = \pm 1. \end{aligned}\right\} \qquad \ldots \quad (7)$$

For every particular pair i, j, the m integers $q_1, \ldots q_m$ have no common factor.

From (5) it is evident that

$$A_{ij}^{e_1 \ldots e_i} = (a_i, a_j)^{e_1 \ldots e_i} = (c, a_j) = 1, \ (i < j)$$

since $a_i^{e_1 \ldots e_i} = c$ is an element of C.

Now from (7) one gets

$$A_{1i}^{e_1} = b_1^{e_1 q_1} b_2^{e_1 q_2} \ldots b_m^{e_1 q_m} = 1 \qquad \ldots \quad (8)$$

so that

$$\left.\begin{array}{lll} e_1 q_1 & \text{is divisible by} & f_1 \\ e_1 q_2 & \ldots & f_1 f_2 \\ e_1 q_m & \ldots & f_1 f_2 \ldots f_m \end{array}\right\} \qquad \ldots \quad (9)$$

Since the q's have got no common factor, e_1 is divisible by f_1. Let $e_1 = f_1 e'_1$.

Again by taking from the second term of the right hand side of (9), one sees that the $m-1$ numbers $e'_1 q_2, \ldots e'_1 q_m$ are all divisible by f_2. So either $q_2, \ldots q_m$ have got a factor in common with f_2 or e_1 is divisible by all the prime factors of f_2. If t is a common prime factor of f_2, $q_2, \ldots q_m$, but not contained in f_1, then there exists an element b such that $b^t = b_1$ and $b, b_2, \ldots b_m$ are a system of generators of C. For this system, the corresponding numbers $q_1, q_2 \ldots q_m$ are divisible by t and that is impossible. Hence e'_1 is divisible by t, and therefore $e_1 = f_1 e'_1$ is divisible by every prime factor of f_2.

The same method can be applied to show that e_1 is divisible by every prime factor of $f_1, f_2, \ldots f_m$. Suppose it is proved for the prime factors of f_j, $j<k$. Then $e_1 q_k$, $e_1 q_{k+1}, \ldots e_1 q_m$ are divisible by f_k. Let u be a common prime factor of f_k, $q_k, \ldots q_m$, but not a factor of $f_1 f_2 \ldots f_{k-1}$. Then there exist elements $d_1, \ldots d_{k-1}$, such that $d_j^u = b_j$, $j<k$, and using $d_1, d_2, \ldots d_{k-1}, b_k, \ldots b_m$ as a system of generators of C, one obtains a system of q's, $q_1, \ldots q_m$ which have u as a common factor, and by the method used in connection with mathematical induction above, one concludes:

e_1 is divisible by every prime factor of $f_1, f_2, \ldots f_m$ and it is divisible by f_1.

It follows therefore that every S-group with n generators can be represented in the form

$$a_1^{p_1} \ldots a_n^{p_n} b_1^{s_1} \ldots b_m^{s_m},$$

with the relations (5) and (6) and commutators being represented by (7). The integers $e_1, e_2 \ldots e_n$, and $f_1, f_2, \ldots f_m$ are invariants of the groups. If any $e=0$, then all the following e's are equal to zero and correspondingly for the f's. The e's and f's are not independent, but are connected by the conditions given above. If in particular, $f_1 = 0$, *i.e.*, if all the elements, different from the unit element of the commutator group, are of infinite order, then $e_1 = 0$, and the group is a free S-group

In conclusion, I express my thanks to Prof. F. W. Levi, of Calcutta University, for his kind guidance and assistance throughout the paper.

DEPARTMENT OF PURE MATHEMATICS
CALCUTTA UNIVERSITY.

Reference

LEVI, F. W., 1942, *Jour. Ind. Math. Soc.*, 6.

All correspondence with the Society, subscriptions to the Bulletin, admission fees and annual contributions of members are to be sent to the *Secretary, Calcutta Mathematical Society, 92, Upper Circular Road, Calcutta.*

Papers intended for publication in the Bulletin of the Society, and all Editorial Correspondence, should be addressed to the *Editorial Secretary, Calcutta Mathematical Society.*

The publications of the Calcutta Mathematical Society may be purchased direct from the Society's office, or from its agents—Messrs. Bowes & Bowes, Booksellers and Publishers, 1, Trinity Street, Cambridge, England.

NOTICE TO AUTHORS

The manuscript of each paper communicated for publication in this Journal should be legibly written (preferably type-written) on one side of the paper and should be accompanied by a short *abstract* of the paper.

References to literature in the text should be given, whenever possible, in chronological order, only the names of the authors and years of publication, in brackets, being given. They should be cited in full at the end of the paper, the author's name following in alphabetical order, and each reference is to be arranged in the following form, *viz.*, name or names of authors ; year of publication ; name of the journal (abbreviation) ; number of volume ; and lastly, the page number. The following would be a useful illustration :

Wilson, B. M., 1922, *Proc. Lond. Math. Soc.* (2), 21, 235.

Authors of papers printed in the Bulletin are entitled to receive, free of cost, 25 separate copies of their communications. They can, however, by previous notice to the Secretary, ascertain whether it will be possible to obtain more copies even on payment of the usual ges.

BULLETIN

OF THE

CALCUTTA MATHEMATICAL SOCIETY

A limited number of back Volumes of the Bulletin of the Calcutta Mathematical Society will be sold at reduced prices.

For particulars write to:—

THE SECRETARY, CALCUTTA MATHEMATICAL SOCIETY,
92, UPPER CIRCULAR ROAD, CALCUTTA.

PRINTED IN INDIA
PRINTED BY DINABANDHU GANGULEE, B.A., SUPERINTENDENT, CALCUTTA UNIVERSITY PRESS, 48, HAZRA ROAD, BALLYGUNGE, CALCUTTA, AND PUBLISHED BY THE CALCUTTA MATHEMATICAL SOCIETY, AT 92 UPPER CIRCULAR ROAD, CALCUTTA.

BULLETIN

OF THE

CALCUTTA

MATHEMATICAL SOCIETY

BOARD OF EDITORS

N. M. Basu, D.Sc.
F. W. Levi, Dr.Phil.Nat.
C. V. H. Rao, M.A.
N. R. Sen, D.Sc., Ph.D.
M. R. Siddiqi, M.A., Ph.D.
N. R. Sen, M.A., Ph.D.
S. Gupta, M.Sc. (*Secretary*)

Vol. 35 December, 1943 No. 4

CONTENTS

PAGE

16. A Matrix-Theory of Screws in Hyper-Space—By N. N. Ghosh ... 115
17. A Note on Sonine's Polynomials (II)—By K. Basu 127
18. On a Problem in Heat Conduction in a Sphere—By N. R. Sen ... 131
19. Note on the Uniqueness of Solution of Problems of Thin Plates bent by Normal Pressure—By Bibhutibhusan Sen 135
20. Highly Abundant Numbers—By S. S. Pillai 141

CALCUTTA UNIVERSITY PRESS
1944

All correspondence with the Society, subscriptions to the Bulletin, admission fees and annual contributions of members are to be sent to the *Secretary, Calcutta Mathematical Society, 92, Upper Circular Road, Calcutta.*

Papers intended for publication in the Bulletin of the Society, and all Editorial Correspondence, should be addressed to the *Editorial Secretary, Calcutta Mathematical Society.*

The publications of the Calcutta Mathematical Society may be purchased direct from the Society's office, or from its agents—Messrs. Bowes & Bowes, Booksellers and Publishers, 1, Trinity Street, Cambridge, England.

NOTICE TO AUTHORS

The manuscript of each paper communicated for publication in this Journal should be legibly written (preferably type-written) on one side of the paper and should be accompanied by a short *abstract* of the paper.

References to literature in the text should be given, whenever possible, in chronological order, only the names of the authors and years of publication, in brackets, being given. They should be cited in full at the end of the paper, the author's name following in alphabetical order, and each reference is to be arranged in the following form, *viz.*, name or names of authors ; year of publication ; name of the journal (abbreviation) ; number of volume ; and lastly, the page number. The following would be a useful illustration :

Wilson, B. M., 1922, *Proc. Lond. Math. Soc.* (2), 21, 235.

Authors of papers printed in the Bulletin are entitled to receive, free of cost, 25 separate copies of their communications. They can, however, by previous notice to the Secretary, ascertain whether it will be possible to obtain more copies even on payment of the usual charges.

A MATRIX-THEORY OF SCREWS IN HYPERSPACE

By

N. N. Ghosh

(Received August 24, 1943)

In a recent paper (Ghosh, 1940) it has been shown that a system of forces acting on a 'rigid' body in a Euclidean n-space referred to a given set of mutually orthogonal axes can be represented by means of the general skew-symmetric matrix

$$\mathrm{P}+\Omega, \qquad \ldots \quad (1)$$

where

$$\mathrm{P}=\sum_{s=2}^{n+1} p_{s-1}(e_{s1}-e_{1s}) \qquad \ldots \quad (2)$$

corresponds to the resultant force acting at the origin and

$$\Omega = \sum_{s,t=2}^{n+1} \omega_{s-1,t-1}(e_{st}-e_{ts}), \quad t > s \qquad \ldots \quad (3)$$

corresponds to the resultant couple of the generalized type, e's in the above denoting matrix units.

Further, with regard to the instantaneous motion of a rigid body it has been shown that the velocity $\dot{\mathrm{X}}$ at a point X at time t is expressible by means of the matrix

$$\dot{\mathrm{X}}=\dot{\mathrm{H}}+\Omega\mathrm{X}-\mathrm{X}\Omega \qquad \ldots \quad (4)$$

of type (2), where $\dot{\mathrm{H}}$ corresponds to the velocity of translation and $\Omega\mathrm{X}-\mathrm{X}\Omega$ to 'rotation' about the origin, Ω being represented by (3) as a function of t.

The object of this paper is to discuss the canonical forms to which (1) and (4) can be reduced. In a space of odd dimension both the reductions are associated with the corresponding generalized notion of a screw.* For a space of even dimension, on the other hand, no reduction to a screw motion is in general possible with regard to (4), but with respect to (1) the canonical reduction may be of the screw type having certain peculiarities. For instance, in a four-space a linear three-dimensional space may be assigned such that outside it the canonical form of (1) is not of the screw type, but inside, at a point, the canonical form involves the notion of a screw.

The application of matrices to the theory of screws in hyperspace is believed to be new. An idea of using matrix notation to represent a screw in statics for ordinary space, is, however, contained in a paper by Hudson (1903). Hitting upon the representation of a force-system similar to the matrix (1) and taking into consideration its reciprocal and other combinations of type (1) he obtained some properties relating to a three-dimensional space. The possibility of extending the notation to hyperspace was obviously not in view.

---

* The term 'screw' is used in this paper in a general sense, having either a statical or a dynamical significance.

Some general discussions on the theory of screws in hyperspace are to be found in a paper by Joly (1898) in connection with his development of an Algebra based on the Clifford's units i_1, $i_2, \ldots i_n$ satisfying the equations $i_s^2 = -1$, $i_s i_t + i_t i_s = 0$. With regard to the canonical representation of a screw in Dynamics his discussions are more limited than those in statics. The remark 'Nor do I see how to obtain an expression analogous to the pitch of a wrench' is significant of some incompleteness in his theory.

Isolated problems on the theory of screws in hyperspace have, however, been tackled by few more authors using convenient analysis. Based on a determinantal method Eiesland (1904) analysed the most general infinitesimal motion in a space of n-dimension and (for n odd) arrived at the notion of an n-dimensional screw motion 'consisting of a translation along an axis and $\frac{n-1}{2}$ rotations.' Woinarosky (1935) considered the problem of compounding together two instantaneous rigid motions in a space of n (even) dimension represented by means of bivectors and obtained the relation between the respective centres of rotation.

The matrix method adopted in this paper is comprehensive enough for a systematic treatment of all such problems. It will be noticed that the above-mentioned 'difficulty' experienced by Joly is removed in this exposition. Before taking up the general analysis, it will be of some interest to study cases for a few particular values of n in the light of the general process. The notations and results of my previous paper will be freely used in this connection.

2. Case 1: $n=2$.

The matrix (1) is now of the simple form

$$\begin{bmatrix} 0 & -p_1 & -p_2 \\ p_1 & 0 & \omega_{12} \\ p_2 & -\omega_{12} & 0 \end{bmatrix}, \qquad \ldots (5)$$

which represents a force P and a couple Ω in the plane, determined by the pair of mutually orthogonal unit vectors $e_{21} - e_{12}$ and $e_{31} - e_{13}$.
Introducing a vector represented by

$$\mathrm{A} = a_1(e_{21} - e_{12}) + a_2(e_{31} - e_{13})$$

orthogonal to P we can express Ω in (5) in the form

$$\Omega = \mathrm{AP} - \mathrm{PA}.$$

Thus when $n=2$, the required canonical form of (1) is

$$\mathrm{P} + \mathrm{AP} - \mathrm{PA}, \qquad \ldots (6)$$

which represents a force P acting at the point A.

To determine the components of A we have the equations

$$\left.\begin{aligned} a_1p_1+a_2p_2&=0\\ -a_1p_2+a_2p_1&=\omega_{12}\end{aligned}\right\} \qquad \ldots \quad (7)$$

corresponding to the dynamical equation (4), let us change the base point to Y, the new equation then appears in the form

$$\dot{X}=\dot{\bar{H}}+\Omega(X-Y)-(X-Y)\Omega, \qquad \ldots \quad (8)$$

where

$$\dot{\bar{H}}=\dot{H}+\Omega Y-Y\Omega. \qquad \ldots \quad (9)$$

when $n=2$, for a canonical reduction we choose Y in (8) such that the velocity of translation

$$\dot{\bar{H}}=0.$$

This gives the equations

$$\left.\begin{aligned} \dot{h}_1+\omega_{12}y_2&=0\\ \dot{h}_2-\omega_{12}y_1&=0\end{aligned}\right\} \qquad \ldots \quad (10)$$

to determine Y, the centre of instantaneous rotation.

3. Case 2: $n=3$.

We first observe that the matrix Ω in (1) with elements $(\omega_{12}, \omega_{13}, \omega_{23})$ may be expressed in any of the forms

$$\Omega=-\frac{M_1M_2-M_2M_1}{\omega_{12}}=-\frac{M_2M_3-M_3M_2}{\omega_{23}}=-\frac{M_1M_3-M_3M_1}{\omega_{13}},$$

where

$$\left.\begin{aligned} M_1&=\omega_{12}(e_{31}-e_{13})+\omega_{13}(e_{41}-e_{14}),\\ M_2&=-\omega_{12}(e_{21}-e_{12})+\omega_{23}(e_{41}-e_{14}),\\ M_3&=-\omega_{13}(e_{21}-e_{12})-\omega_{23}(e_{31}-e_{13}),\end{aligned}\right\} \qquad \ldots \quad (11)$$

The above vectors are *associated* with the couple Ω and are connected by the equation

$$M_{123}=M_1\omega_{23}+M_2\omega_{31}+M_3\omega_{12}=0. \qquad \ldots \quad (12)$$

It may be noted that the vector

$$\omega_{23}(e_{21}-e_{12})-\omega_{13}(e_{31}-e_{13})+\omega_{12}(e_{41}-e_{14})$$

is orthogonal to the set (11) and represents the *axis* of the couple. For a canonical reduction to the screw type we write (1) in the form

$$P+AP-PA+\bar{\Omega}\left\{\begin{matrix}A\\P\end{matrix}\right\}=0, \qquad \ldots \quad (13)$$

where

$$\bar{\Omega}=\Omega-AP+PA \qquad \ldots \quad (14)$$

and choose A such that P is orthogonal to the plane of the couple $\bar{\Omega}$.

To determine A we have then the equations

$$\left.\begin{aligned} p_2\bar{\omega}_{12}+p_3\bar{\omega}_{13}&=0,\\ p_1\bar{\omega}_{12}-p_3\bar{\omega}_{23}&=0,\\ a_1p_1+a_2p_2+a_3p_3&=0,\end{aligned}\right\} \quad \ldots \quad (15)$$

where

$$\bar{\omega}_{ij}=\omega_{ij}+a_ip_j-p_ia_j.$$

For a canonical reduction of (4) to the screw type we take (8) and choose the base point Y such that $\dot{\bar{H}}$ is orthogonal to the plane of the couple Ω. When $n=3$, this gives the equations

$$\frac{\dot{h}_1+\omega_{12}y_2+\omega_{13}y_3}{\omega_{23}}=\frac{\dot{h}_2+\omega_{23}y_3-\omega_{12}y_1}{\omega_{31}}=\frac{\dot{h}_3-\omega_{13}y_1-\omega_{23}y_2}{\omega_{12}}, \quad \ldots \quad (16)$$

which shows that the components of Y generate a line, the so-called central axis.

4. Case 3: $n=4$.

Associated with Ω we have now the set of 4 vectors

$$\left.\begin{aligned} M_1&=\omega_{12}(e_{31}-e_{13})+\omega_{13}(e_{41}-e_{14})+\omega_{14}(e_{51}-e_{15}),\\ M_2&=-\omega_{12}(e_{21}-e_{12})+\omega_{23}(e_{41}-e_{14})+\omega_{24}(e_{51}-e_{15}),\\ M_3&=-\omega_{13}(e_{21}-e_{12})-\omega_{23}(e_{31}-e_{13})+\omega_{34}(e_{51}-e_{15}),\\ M_4&=-\omega_{14}(e_{21}-e_{12})-\omega_{24}(e_{31}-e_{13})-\omega_{34}(e_{41}-e_{14}).\end{aligned}\right\} \quad \ldots \quad (17)$$

If (i, j, k, l) denote the sequence (1, 2, 3, 4) in some order, it may be shown that Ω is expressible in the form

$$\Omega=-\frac{M_iM_j-M_jM_i}{\omega_{ij}}+\frac{\omega_{ijkl}}{\omega_{ij}}(e_{k+1,l+1}-e_{l+1,k+1})\omega_{ij}\neq 0, \quad \ldots \quad (18)$$

where

$$\omega_{ijkl}=\omega_{ij}\omega_{kl}-\omega_{ik}\omega_{jl}+\omega_{il}\omega_{jk}$$

It must be noted that if Ω be of rank 2, $\omega_{ijkl}=0$ and (18) reduces to

$$\Omega=-\frac{M_iM_j-M_jM_i}{\omega_{ij}}, \quad \ldots \quad (19)$$

the component couple in the co-ordinate plane having vanished.

For a canonical reduction of (1) let us consider the matrix (13). We observe that $\bar{\Omega}$ is of rank 2 and expressible in the form (19) if its components satisfy the equation

$$\bar{\omega}_{1234}=0$$

or

$$\omega_{1234}+a_1p_{234}-a_2p_{134}+a_3p_{124}-a_4p_{123}=0, \quad \ldots \quad (20)$$

where

$$p_{ijk}=p_i\omega_{jk}+p_j\omega_{ki}+p_k\omega_{ij}.$$

Thus for some point A on the linear subspace given by (20), P may be chosen orthogonal to the plane of the couple $\bar{\Omega}$, resulting into a screw.

For all points A not satisfying (20), the canonical reduction of (1) is equivalent to a force P acting at the point combined with a pair of couples represented in (18). By applying (6) this again is equivalent to a pair of forces acting at two definite points.

Next, for the dynamical equation (8) we observe that the linear velocity $\dot{\bar{H}}$ varies with Y, while $\dot{\Omega}$ remains invariable. For $n=4$, no screw motion is therefore possible. We may, however, choose Y such that

$$\dot{\bar{H}}=0.$$

Thus the canonical reduction of (4) is a 'rotation' about a definite point compounded from a pair of simple angular velocities.

We may note here generally that the expression

$$\tfrac{1}{2}\sum_{i,j=1}^{n}\omega_{ij}^2 \qquad \dots \quad (21)$$

represents either the square of the moment of the couple or the square of the magnitude of the angular velocity according as Ω occurs in (1) or (4).

5. Let us now proceed to the general resolution of Ω in (8). A typical vector associated with Ω may be denoted by

$$M_i=\sum_{s=2}^{n+1}\omega_{i,s-1}(e_{s1}-e_{1s}). \qquad \dots \quad (22)$$

Choosing any three of them we can form, by linear combination, the set of vectors

$$M_{ijk}=M_i\omega_{jk}+M_j\omega_{ki}+M_k\omega_{ij}=\sum_{s=2}^{n+1}\omega_{ijk,s-1}(e_{s1}-e_{1s}), \qquad \dots \quad (23)$$

where $\omega_{ijk,s-1}$ denotes the Pfaffian $\omega_{ij}\omega_{k,s-1}-\omega_{ik}\omega_{j,s-1}+\omega_{i,s-1}\omega_{jk}$ of order 4. It may be verified that connecting (22) and (23) there exists the relation

$$M_x\omega_{ijkl}-M_i\omega_{xjkl}=M_{xij}\omega_{kl}-M_{xik}\omega_{jl}+M_{xil}\omega_{jk}-M_{jkl}\omega_{xi}, \quad \dots \quad (24)$$

whence a particular M_x of type (22) can be expressed linearly in terms of three suitably-chosen vectors of the set (23).

Generally, with $(2r+1)$ vectors of the set (22) we can form the set

$$M_{i_1i_2\cdots i_{2r+1}}=M_{i_1}\omega_{i_2i_3\cdots i_{2r+1}}+M_{i_2}\omega_{i_3i_4\cdots i_{2r+1}i_1}+\cdots$$
$$+M_{i_{2r+1}}\omega_{i_1i_2\cdots i_{2r}}, \quad \dots \quad (25)$$

where the ω's are Pfaffians of order $2r$. The above admits of being written in the compact form of a matrix

$$M_{i_1i_2\cdots i_{2r+1}}=\sum_{s=2}^{n+1}\omega_{i_1i_2\cdots i_{2r+1},s-1}(e_{s1}-e_{1s}), \qquad \dots \quad (26)$$

where the elements are Pfaffians of order $2r+2$.

Performing now the operation $M_iM_j - M_jM_i$ involving a pair of vectors of the set (22) we get

$$\frac{M_iM_j - M_jM_i}{\omega_{ij}} = \Omega^{(2)} - \Omega, \qquad \dots \quad (27)$$

where

$$\Omega^{(2)} = \sum_{s,t=2}^{n+1} \frac{\omega_{ij,s-1,t-1}}{\omega_{ij}} (e_{st} - e_{ts}), \quad t > s \qquad \dots \quad (28)$$

A similar combination of a pair of vectors of the set (23) yields the result

$$\frac{M_{ijk}M_{ijl} - M_{ijl}M_{ijk}}{\omega_{ij}\omega_{ijkl}} = \Omega^{(3)} - \Omega^{(2)}, \qquad \dots \quad (29)$$

where

$$\Omega^{(3)} = \sum_{s,t=2}^{n+1} \frac{\omega_{ijkl,s-1,t-1}}{\omega_{ijkl}} (e_{st} - e_{ts}) \qquad \dots \quad (30)$$

By means of an identity due to Brill (1902), it can be shown generally that

$$\frac{\left[M_{i_1i_2\cdots i_{2r-2}i_{2r-1}} M_{i_1i_2\cdots i_{2r-2}i_{2r}}\right]}{\omega_{i_1i_2\cdots i_{2r-2}}\omega_{i_1i_2\cdots i_{2r}}} = \Omega^{(r+1)} - \Omega^{(r)}, \qquad \dots \quad (31)$$

where

$$\Omega^{(r+1)} = \sum_{s,t=2}^{n+1} \frac{\omega_{i_1i_2\cdots i_{2r},s-1,t-1}}{\omega_{i_1i_2\cdots i_{2r}}} (e_{st} - e_{ts}), \qquad \dots \quad (32)$$

the symbol $[MN]$ in the above denoting the combination $MN - NM$.
Let Ω be of rank $2r$, then $\Omega^{(r+1)}$ will vanish and we have

$$\Omega = (\Omega - \Omega^{(2)}) + (\Omega^{(2)} - \Omega^{(3)}) + \dots + (\Omega^{(r-1)} - \Omega^{(r)}) + \Omega^{(r)}.$$

Thus the generalized matrix Ω of rank $2r$ is expressible* as a sum of r simple matrices of the type $MN - NM$.

6. We next consider the matrix $\overline{\Omega}$ in (14). It may be proved by induction that the Pfaffians connected with $\overline{\Omega}$ are all linear functions of the components of A. Denoting a Pfaffian of order $2r$ by $\overline{\omega}_{i_1i_2\cdots i_{2r}}$ we get, in fact,

$$\overline{\omega}_{i_1i_2\cdots i_{2r}} = \omega_{i_1i_2\cdots i_{2r}} + a_{i_1}p_{i_2i_3\cdots i_{2r}} - a_{i_2}p_{i_1i_3\cdots i_{2r}} + \dots$$

$$- a_{i_{2r}}p_{i_1i_2\cdots i_{2r-1}}, \qquad \dots \quad (33)$$

where

$$p_{k_1k_2\cdots k_{2r-1}} = p_{k_1}\omega_{k_2k_3\cdots k_{2r-1}} + p_{k_2}\omega_{k_3k_4\cdots k_{2r-1}k_1} + \dots$$

$$+ p_{k_{2r-1}}\omega_{k_1k_2\cdots k_{2r-2}}. \qquad \dots \quad (34)$$

It may be noted that (34) is equivalent to the expression

$$p_{k_1k_2\cdots k_{2r-1}} = p_{k_1}\omega_{k_2k_3\cdots k_{2r-1}} + \omega_{k_1k_2}p_{k_3k_4\cdots k_{2r-1}} - \omega_{k_1k_3}p_{k_4k_5\cdots k_2} + \dots$$

$$- \omega_{k_1k_{2r-1}}p_{k_2k_3\cdots k_{2r-2}}, \qquad \dots \quad (35)$$

where the suffixes can be cyclically interchanged.

---

* This is a fuller treatment than what was given in my previous paper.

Further, the quantities $p_{k_1k_2\cdots k_{2r-1}}$ satisfy the following identity which can be immediately verified.

$$p_{k_1}p_{k_2k_3\cdots k_{2r}}-p_{k_2}p_{k_1k_3\cdots k_{2r}}+\cdots-p_{k_{2r}}p_{k_1k_2\cdots k_{2r-1}}=0. \quad \ldots \quad (36)$$

By means of the above relations the required proof will follow easily.

Observing that (25) is analogous to (34) in form, an equivalent expression for (25) may also be obtained by means of (35).

7. Let us now turn to the equation (13) to deal with the general case. We observe that for a screw the force P will be orthogonal to the *planes* of the couple $\bar{\Omega}$ if

$$\bar{\Omega}\mathrm{P}-\mathrm{P}\bar{\Omega}=0. \quad \ldots \quad (37)$$

When n is even a necessary condition $\bar{\omega}_{12\ldots n}=0$, which is linear in a's, is involved in the above.

Writing (37) in the form

$$\Omega\mathrm{P}-\mathrm{P}\Omega+\mathrm{P}(\mathrm{AP}-\mathrm{PA})-(\mathrm{AP}-\mathrm{PA})\mathrm{P}=0$$

and applying the product formula for matrices of type P, we get

$$\Omega\mathrm{P}-\mathrm{P}\Omega+\left\{\begin{matrix}\mathrm{P}\\\mathrm{P}\end{matrix}\right\}\mathrm{A}-\left\{\begin{matrix}\mathrm{A}\\\mathrm{P}\end{matrix}\right\}\mathrm{P}=0.$$

Remembering $\left\{\begin{matrix}\mathrm{A}\\\mathrm{P}\end{matrix}\right\}=0$, the above gives

$$\mathrm{A}=-\frac{1}{\left\{\begin{matrix}\mathrm{P}\\\mathrm{P}\end{matrix}\right\}}(\Omega\mathrm{P}-\mathrm{P}\Omega). \quad \ldots \quad (38)$$

The matrix $\bar{\Omega}$ is now of the form

$$\Omega+\frac{1}{\left\{\begin{matrix}\mathrm{P}\\\mathrm{P}\end{matrix}\right\}}(\Omega\mathrm{P}^2+\mathrm{P}^2\Omega) \quad \ldots \quad (39)$$

in which a typical element may be written

$$\frac{1}{\left\{\begin{matrix}\mathrm{P}\\\mathrm{P}\end{matrix}\right\}}\sum_{k=1}^{n}p_k p_{ijk}. \quad \ldots \quad (40)$$

Hence the square of the moment of the couple $\bar{\Omega}$, as defined in (21), is given by

$$\frac{1}{2\left\{\begin{matrix}\mathrm{P}\\\mathrm{P}\end{matrix}\right\}^2}\sum_{i,j,k,l=1}^{n}p_k p_l p_{ijk} p_{ijl} \quad \ldots \quad (41)$$

By means of the identical relation

$$p_l p_{ijk}-p_i p_{ljk}+p_j p_{lik}-p_k p_{lij}=0$$

noted in (36), the above may be reduced to the form

$$\frac{\Sigma_c p_{ijk}^2}{\sum_{i=1}^{n} p_i^2}, \qquad \ldots \quad (42)$$

where Σ_c indicated that the summation extends over all combinations (ijk) of the numbers 1, 2,...n.

For the axis of the screw we have the equation

$$S = A + xP, \qquad \ldots \quad (43)$$

where x is a scalar variable.

It may be noted that the typical form of a screw at a point A is given by

$$P + AP - PA + \Omega, \qquad \ldots \quad (44)$$

where $$\Omega P - P\Omega = 0, \quad \left\{ \begin{matrix} A \\ P \end{matrix} \right\} = 0. \qquad \ldots \quad (45)$$

Let us now proceed to determine the screw compounded from two given screws of the above type at points A′ and A″. Let P′, Ω′ correspond to the former and P″, Ω″ to the latter. If now (44) represents the screw compounded from the above we have by applying (38)

$$-\left\{ \begin{matrix} P \\ P \end{matrix} \right\} A = q'q''\left[\Omega'P'' - P''\Omega' + \Omega''P' - P'\Omega'' + \left\{ \begin{matrix} A' \\ P'' \end{matrix} \right\} P' + \left\{ \begin{matrix} A'' \\ P' \end{matrix} \right\} P'' \right]$$
$$-\left(q'^2 \left\{ \begin{matrix} P' \\ P' \end{matrix} \right\} + q'q'' \left\{ \begin{matrix} P' \\ P'' \end{matrix} \right\} \right) A' - \left(q''^2 \left\{ \begin{matrix} P'' \\ P'' \end{matrix} \right\} + q'q'' \left\{ \begin{matrix} P' \\ P'' \end{matrix} \right\} \right) A'', \qquad \ldots \quad (46)$$

where q's arbitrary scalars so that

$$P = q'P' + q''P''.$$

The locus of the axes represented by

$$S = A + x(q'P' + q''P'') \qquad \ldots \quad (47)$$

is the analogue of the *cylindroid.*

If D denote the shortest vector between the given screws, the origin may be chosen on this line and the expression for A, given in (46), is much simplified. It is easy to see that the vector represented by $\Omega'P'' - P''\Omega' + \Omega''P' - P'\Omega''$ is also orthogonal to P′ and P″. The point A thus lies in the plane of D and this vector, that is, a plane which is *completely orthogonal* to the plane of P′ and P″.

An ordinary analytical method of studying the above in a 3-dimensional space is given in a paper by Srinivasiengar (1942). It may be noted that p_{ijk} corresponding to the compounded screw is given by the expression

$$q'^2 p'_{ijk} + q''^2 p''_{ijk} + q'q''[\Sigma_0 p'_i \omega''_{jk} + \Sigma_0 p''_i \omega'_{jk} - \Sigma_0 (p'_i p''_j - p''_i p'_j)(a'_k - a''_k)], \qquad \ldots \quad (48)$$

where Σ_0 indicates that the summation is to be performed cyclically over the indices i, j, k.

8. For a canonical reduction of (4) to the screw type we consider the equation (8) and take n to be an odd integer. We observe that $\dot{\overline{\overline{\mathrm{H}}}}$ is orthogonal to the planes of Ω if

$$\Omega\dot{\overline{\overline{\mathrm{H}}}}-\dot{\overline{\overline{\mathrm{H}}}}\Omega=0$$

or $$\Omega\dot{\mathrm{H}}-\dot{\mathrm{H}}\Omega+\Omega^2\mathrm{Y}+\mathrm{Y}\Omega^2=0, \qquad \ldots \quad (49)$$

whence Y is to be obtained.

Let the characteristic equation satisfied by Ω be written in the form

$$f(\Omega).\Omega^2=\Omega, \qquad \ldots \quad (50)$$

where $f(\Omega)$ is a rational integral function of Ω. Then evidently

$$-\mathrm{Y}=f(\Omega)\dot{\mathrm{H}}-\dot{\mathrm{H}}f(\Omega) \qquad \ldots \quad (51)$$

is a solution of (49).

Corresponding to it, $\dot{\overline{\overline{\mathrm{H}}}}$ has the value

$$\dot{\mathrm{H}}-\Omega f(\Omega)\dot{\mathrm{H}}-\dot{\mathrm{H}}f(\Omega)\Omega. \qquad \ldots \quad (52)$$

Generally for all points on the line $\mathrm{S}=\mathrm{Y}+x\dot{\overline{\overline{\mathrm{H}}}}$, where Y is given by (51), the condition for a screw motion is satisfied.

Let u_1 denote the elementary matrix

$$\sum_{s=2}^{n+1} e_{ss},$$

then the square of the magnitude of the translational velocity $\dot{\overline{\overline{\mathrm{H}}}}$ is given by the matrix of type e_{11}:

$$\dot{\overline{\overline{\mathrm{H}}}}\,u_1\dot{\overline{\overline{\mathrm{H}}}},$$

which by means of (52) reduces* to

$$\dot{\mathrm{H}}(u_1-f(\Omega)\Omega)\dot{\mathrm{H}}. \qquad \ldots \quad (53)$$

It may be noted that the typical form of a screw corresponding to (4) may be expressed as

$$\dot{\mathrm{X}}=\dot{\mathrm{H}}+\Omega(\mathrm{X}-\mathrm{Y})-(\mathrm{X}-\mathrm{Y})\Omega, \qquad \ldots \quad (54)$$

where $$\Omega\dot{\mathrm{H}}-\dot{\mathrm{H}}\Omega=0,\quad \begin{Bmatrix}\mathrm{Y}\\ \dot{\mathrm{H}}\end{Bmatrix}=0,$$

which is analogous to (44).

When we proceed to form the screw compounded from a pair of screws of the type (54), it appears that the equation corresponding to (46) is more complicated and the usual analogy which persists in 3-dimensional space does not hold good generally.

---

* This supplies 'an expression analogous to pitch of a wrench' sought by Joly, *loc. cit.*, p. 118.

If n is even and Ω undegenerate as a canonical reduction of (4) we have to choose $\ddot{H}=0$ in (8). This gives the equation

$$\dot{H}+\Omega Y-Y\Omega=0 \qquad \dots \quad (55)$$

to determine Y. Thus the general infinitesimal motion consists of rotations about a fixed point in $n/2$ planes, defined by the component couples of Ω.

9. We conclude this paper by generalizing what is called the 'virtual coefficient' of a pair of screws (Ball, 1900).

Let the general infinitesimal displacement at X of a rigid body be represented by the screw of type (54):

$$\delta X=\delta H+\delta\Lambda(X-Y)-(X-Y)\delta\Lambda, \qquad \dots \quad (56)$$

where

$$\delta\Lambda.\delta H-\delta H.\delta\Lambda=0, \quad \left\{\begin{matrix} Y \\ \delta H \end{matrix}\right\}=0,$$

and the force-system by means of the screw of type (44):

$$P+AP-PA+\Omega, \qquad \dots \quad (57)$$

where

$$\Omega P-P\Omega=0, \quad \left\{\begin{matrix} A \\ P \end{matrix}\right\}=0.$$

First we observe that the work done by a force P acting at A due to the displacement δA is given by

$$-Pu_1\delta A,$$

which by means of (56) becomes

$$-Pu_1\delta H-P.\delta\Lambda.(A-Y) \qquad \dots \quad (58)$$

Combining this with the work done by a force $-P$ at B due to the displacement δB at the point we get the work done by the couple $(A-B)P-P(A-B)$ in the form

$$(A-B).\delta\Lambda.P. \qquad \dots \quad (59)$$

By evaluating the matrix products involved in the above, it follows that the work done by any couple Γ may be expressed in the symmetrical form

$$e_{11}\sum_{i,j=1}^{n}\delta\lambda_{ij}\gamma_{ij}, \quad i<j, \qquad \dots \quad (60)$$

where $\delta\lambda_{ij}$ and γ_{ij} are the corresponding elements of $\delta\Lambda$ and Γ. The expression (60) at once applies to a system of couples and hence the work done by the generalized couple Ω in (57) is represented by

$$e_{11}\sum_{i,j=1}^{n}\delta\lambda_{ij}\omega_{ij}, \quad i<j \qquad \dots \quad (61)$$

The total work done by the force-system (57) is therefore obtained as the sum of (58) and (61).

Now the product $-\mathrm{P}.\delta\Lambda.(\mathrm{A}-\mathrm{Y})$ in (58) is equivalent to

$$\mathrm{D}.\delta\Lambda.\mathrm{P},$$

where D denotes the shortest vector between the pair of screws expressed as

$$\mathrm{A}-\mathrm{Y}+x\mathrm{P}+y\delta\mathrm{H},$$

x, y being scalars.

Hence, introducing the couple

$$\Gamma=\mathrm{DP}-\mathrm{PD},$$

with elements $\gamma_{i,j}$, (58) may be written in the form

$$\begin{Bmatrix} \mathrm{P} \\ \delta\mathrm{H} \end{Bmatrix} e_{11}+e_{11}\sum_{i,j=1}^{n}\delta\lambda_{i,j}\gamma_{i,j}, \quad i<j \qquad \ldots \quad (62)$$

The total quantity of work is, therefore,

$$\begin{Bmatrix} \mathrm{P} \\ \delta\mathrm{H} \end{Bmatrix} + \sum_{i,j=1}^{n} \delta\lambda_{i,j}(\gamma_{i,j}+\omega_{i,j}), \quad i<j, \qquad \ldots \quad (62)$$

which is the required generalized expression for the 'virtual coefficient.'

References

Ball, R. S. (1900), *Theory of Screws*, p. 17.
Brill, J. (1902), *Quarterly Journal*, 33, 267.
Eiesland, J. (1904), *Amer. J. of Math.*, 26, 112.
Ghosh, N. N. (1940), *Bull. Cal. Math. Soc.*, 32, 118.
Hudson, R. W. T. H. (1903), *Mess. Math.*, 32, 51.
Joly, C. J. (1898), *Proc. Roy. Irish. Acad.* (3) 5, 111.
Srinivasiengar (1942), *Bull. Cal. Math. Soc.*, 34, 133.
Woinarosky, R. (1935); *Rend. R. Accad. Naz. dei Lincei*, 22, 34.

A NOTE ON SONINE'S POLYNOMIALS * (II)

By

K. Basu

(Communicated by the Secretary—Received September 25, 1943)

I

$$\text{Since}\quad \frac{\exp.-(z_1+z_2)t./1-t}{(1-t)^{\lambda+\mu}}=\frac{\exp.(-z_1t/1-t)}{(1-t)^{\lambda}}\cdot\frac{\exp.(-z_2t/1-t)}{(1-t)^{\mu}}\quad\begin{pmatrix}\lambda>0\\ \mu>0\\ |t|<1\end{pmatrix}$$

$$\therefore \sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{\Gamma(\lambda+\mu+k)}{\Gamma(\lambda+\mu)}\mathrm{S}^{k}_{\lambda+\mu}(z_1+z_2)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\mathrm{S}^{n}_{\lambda}(z_1)\sum_{m=0}^{\infty}\frac{t^m}{m!}\frac{\Gamma(\mu+m)}{\Gamma(\mu)}\mathrm{S}^{m}_{\mu}(z_2)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^{n+m}}{n!m!}\frac{\Gamma(\lambda+n)\Gamma(\mu+m)}{\Gamma(\lambda)\Gamma(\mu)}\mathrm{S}^{n}_{\lambda}(z_1)\mathrm{S}^{m}_{\mu}(z_2).$$

If $n+m=k$ we get equating coefficients of t^k from both sides:

$$\mathrm{S}^{k}_{\lambda+\mu}(z_1+z_2)=\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)\Gamma(\mu)}\sum_{\tau=0}^{k}\frac{k!}{\tau!(k-\tau)!}\frac{\Gamma(\lambda+\tau)\Gamma(\mu+k-\tau)}{\Gamma(\lambda+\mu+k)}\mathrm{S}^{\tau}_{\lambda}(z_1)\mathrm{S}^{k-\tau}_{\mu}(z_2). \quad (1.1)$$

Whence, for $z_1=z_2=z$:

$$\mathrm{S}^{k}_{\lambda+\mu}(2z)=\frac{k!\Gamma(\lambda+\mu)}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\lambda+\mu+k)}\sum_{\tau=0}^{k}\frac{\Gamma(\lambda+\tau)\Gamma(\mu+k-\tau)}{\tau!(k-\tau)!}\mathrm{S}^{\tau}_{\lambda}(z)\mathrm{S}^{k-\tau}_{\mu}(z). \quad \ldots \quad (1.2)$$

Also, for $z_1=x+y$, $z_2=x-y$ $(x>y)$

$$\mathrm{S}^{k}_{\lambda+\mu}(2x)=\frac{k!\Gamma(\lambda+\mu)}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\lambda+\mu+k)}\sum_{\tau=0}^{k}\frac{\Gamma(\lambda+\tau)\Gamma(\mu+k-\tau)}{\tau!(k-\tau)!}\mathrm{S}^{\tau}_{\lambda}(x+y)\mathrm{S}^{k-\tau}_{\mu}(x-y). \quad (1.3)$$

If, further $\lambda=\mu=\nu$, and $z_1=z_2=z$

$$\mathrm{S}^{k}_{2\nu}(2z)=\frac{k!\Gamma(2\nu)}{\Gamma(\nu)\Gamma(\nu)\Gamma(k+2\nu)}\sum_{\tau=0}^{k}\frac{\Gamma(\nu+\tau)\Gamma(\nu+k-\tau)}{\tau!(k-\tau)!}\mathrm{S}^{\tau}_{\nu}(z)\mathrm{S}^{k-\tau}_{\nu}(z). \quad \ldots \quad (1.4)$$

II.

Since $\mathrm{S}\equiv\mathrm{S}^{n}_{\lambda}(\lambda_1 r)$ satisfies [See I]

$$r\frac{d^2\mathrm{S}}{dr^2}+(\lambda-\lambda_1 r)\frac{d\mathrm{S}}{dr}+n\lambda_1\mathrm{S}=0,$$

and $$\frac{d^2}{dz^2}\left(e^{-\frac{1}{2}z}z^{\frac{1}{2}\lambda}\mathrm{S}(z)\right)=e^{-\frac{1}{2}z}z^{\frac{1}{2}\lambda}\left[\frac{d^2\mathrm{S}(z)}{dz^2}+\left(\frac{\lambda}{z}-1\right)\frac{d\mathrm{S}(z)}{dz}+\zeta\mathrm{S}(z)\right],$$

---

* It is a continuation of a previous paper (1943), which will be referred to as I.

where $\zeta \equiv \frac{1}{4} - \frac{\lambda}{2z} + \frac{\lambda(\lambda-2)}{4z^2}$; we get by putting $\psi^n_\lambda(z) = e^{-\frac{1}{2}z} z^{\frac{\lambda}{2}} S^n_\lambda(z)$

$$\frac{d^2\psi^n_\lambda(z)}{dz^2} + \left\{ -\frac{1}{4} + \left(n + \frac{\lambda}{2}\right)\frac{1}{z} - \frac{\lambda(\lambda-2)}{4z^2} \right\} \psi^n_\lambda(z) = 0,$$

and, consequently $z = \lambda_1 r$ gives

$$\frac{d^2\psi^n_{\lambda,\lambda_1}}{dr^2} + \left\{ -\frac{1}{4}\lambda_1^2 + \left(n + \frac{\lambda}{2}\right)\frac{\lambda_1}{r} - \frac{\lambda(\lambda-2)}{4r^2} \right\} \psi^n_{\lambda,\lambda_1} = 0, \qquad \ldots \quad (2.1)$$

where $\psi^n_{\lambda,\lambda_1}(r) \equiv e^{-\frac{1}{2}\lambda_1 r} (\lambda_1 r)^{\frac{\lambda}{2}} S^n_\lambda(\lambda_1 r) = \psi$ say.

Similarly, $\psi' = \psi^n_{\lambda,\lambda_2}(r) \sim e^{-\frac{1}{2}\lambda_2 r} r^{\frac{\lambda}{2}} S^n_\lambda(\lambda_2 r)$ satisfies

$$\frac{d^2\psi'}{dr^2} + \left\{ -\frac{1}{4}\lambda_2^2 + \left(n + \frac{\lambda}{2}\right)\frac{\lambda_2}{r} - \frac{\lambda(\lambda-2)}{4r^2} \right\} \psi' = 0. \qquad \ldots \quad (2.2)$$

Multiply (2.1), (2.2) by ψ', ψ respectively, subtract the results, and get on integrating between 0 and ∞ :

$$\frac{1}{\lambda_1 - \lambda_2} \left| \psi' \frac{d\psi}{dr} - \psi \frac{d\psi'}{dr} \right|_0^\infty = \int_0^\infty \left[\frac{1}{4}\left(\lambda_1 + \lambda_2\right) - \left(n + \frac{\lambda}{2}\right)\frac{1}{r} \right] \psi\psi' dr.$$

Since the integrated part vanishes at both limits, we have (unless $\lambda_1 = \lambda_2$) :

$$\int_0^\infty \left[\frac{1}{4}\left(\lambda_1 + \lambda_2\right) - \left(n + \frac{\lambda}{2}\right)\frac{1}{r} \right] e^{-\frac{1}{2}(\lambda_1 + \lambda_2)r} r^\lambda S^n_\lambda(\lambda_1 r) S^n_\lambda(\lambda_2 r) dr = 0 ;$$

which means $\int_0^\infty r^\lambda e^{-\frac{1}{2}(\lambda_1+\lambda_2)r} S^n_\lambda(\lambda_1 r)\, S^n_\lambda(\lambda_2 r) dr$

$$= \frac{2n+\lambda}{\frac{1}{2}(\lambda_1+\lambda_2)} \int_0^\infty r^{\lambda-1} e^{-\frac{1}{2}(\lambda_1+\lambda_2)r} S^n_\lambda(\lambda_1 r) S^n_\lambda(\lambda_2 r) dr. \qquad \ldots \quad (2.3)$$

III

Now

$$\frac{\exp.(-\lambda_1 rt/1-t)}{(1-t)^\lambda} = \sum_{n=0}^\infty \frac{t^n}{n!} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} S^n_\lambda(\lambda_1 r) \quad (|t| < 1) \qquad \ldots \quad (3.1)$$

$$\frac{\exp.(-\lambda_2 rs/1-s)}{(1-t)^\lambda} = \sum_{m=0}^\infty \frac{s^m}{m!} \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} S^m_\lambda(\lambda_2 r) \quad (|s| < 1) \qquad \ldots \quad (3.2)$$

Multiply both sides of (3.1) by $\exp.(-\frac{1}{2}\lambda_1 r)$, and those of (3.2) by $\exp.(-\frac{1}{2}\lambda_2 r)$, then multiply together the results of each side, and introducing finally a factor $r^{\lambda-1}$ integrate between 0 and ∞. We derive (for $n = m$) :

$$\sum_{n=0}^\infty \frac{(ts)^n}{(n!)^2} \left\{ \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \right\}^2 \int_0^\infty \exp.\{-\tfrac{1}{2}(\lambda_1+\lambda_2)r\} r^{\lambda-1} S^n_\lambda(\lambda_1 r) S^n_\lambda(\lambda_2 r) dr$$

$$= \frac{1}{[(1-t)(1-s)]^\lambda} \int_0^\infty r^{\lambda-1} \exp. -\Lambda r, \qquad \ldots \quad (3.3)$$

where $\Lambda = \frac{1}{2}(\lambda_1+\lambda_2)\left[\frac{1-ts+\theta(t-s)}{(1-t)(1-s)}\right]$ and $\theta=(\lambda_1-\lambda_2)/(\lambda_1+\lambda_2)$, $(\lambda_1 \geq \lambda_2)$.

The righthand member of (3.3) gives after integration

$$\frac{\Gamma(\lambda)}{\{\frac{1}{2}(\lambda_1+\lambda_2)\}^{\lambda}}[1-ts+\theta(t-s)]^{-\lambda}\ .$$

Using multinomial theorem we get the general term of $[\ ]^{-\lambda}$ as

$$\sum_{\beta}\sum_{\gamma}\sum_{\delta}(-)^{r'}\frac{\Gamma(\lambda+r')}{\Gamma(\lambda)\beta!\gamma!\delta!}(\theta t)^{\beta}(-\theta s)^{\gamma}(-ts)^{\delta},\quad (r'=\beta+\gamma+\delta)\quad \ldots\quad (3.4)$$

To find the coefficients of $(ts)^n$ in this summation we make $\beta=\gamma$, so that $\beta+\delta=\gamma+\delta=n$. Hence $r'=2\beta+\delta$, and $\beta=n-\delta$; or, $r'=2n-\delta$, and $r'+\gamma+\delta=r'+\beta+\delta=3n-\delta$. Hence (3.4) takes the form:

$$\sum_{\delta=0}^{n}(-)^{n-\delta}\theta^{2n-2\delta}\frac{\Gamma(\lambda+2n-\delta)}{\Gamma(\lambda)\delta!(n-\delta)!(n-\delta)!}(ts)^n.\quad \ldots\quad (3.5)$$

Hence (3.3) gives, on equating coefficients of $(ts)^n$:

$$\int_0^{\infty} r^{\lambda-1}\text{exp.}\{-\tfrac{1}{2}(\lambda_1+\lambda_2)r\}S_{\lambda}^{n}(\lambda_1 r)S_{\lambda}^{n}(\lambda_2 r)dr$$

$$=\left\{\frac{n!\Gamma(\lambda)}{\Gamma(n+\lambda)}\right\}^2\frac{1}{\{\frac{1}{2}(\lambda_1+\lambda_2)\}^{\lambda}}\sum_{\tau=0}^{n}(-)^{n-\tau}\frac{\Gamma(\lambda+2n-\tau)}{\tau!(n-\tau)!(n-\tau)!}\theta^{2n-2\tau}.\quad \ldots\quad (3.6)$$

In particular, for $\lambda_1=\lambda_2=1$:

$$\int_0^{\infty} r^{\lambda-1}e^{-r}S_{\lambda}^{n}(r)S_{\lambda}^{n}(r)dr=\frac{n![\Gamma(\lambda)]^2}{\Gamma(\lambda+n)}\ ;\ [\text{See I, (1.3)}]$$

because, in this case $\theta=0$ (or, the result does not involve θ), which means, $\tau=$ the single value n.

Result (2.3) gives at once:

$$\int_0^{\infty} r^{\lambda}\text{exp.}(-\tfrac{1}{2}(\lambda_1+\lambda_2)r)S_{\lambda}^{n}(\lambda_1 r)S_{\lambda}^{n}(\lambda_2 r)dr$$

$$=\frac{2n+\lambda}{\{\frac{1}{2}(\lambda_1+\lambda_2\}^{\lambda+1}}\left\{\frac{n!\Gamma(\lambda)}{\Gamma(n+\lambda)}\right\}^2\cdot\sum_{\tau=0}^{n}(-)^{n-\tau}\frac{\Gamma(\lambda+2n-\tau)}{\tau!(n-\tau)!(n-\tau)!}\theta^{2n-2\tau}.\quad \ldots\quad (3.7)$$

In particular, for $\lambda_1=\lambda_2=1$ ($n=\tau$ only) we derive:

$$\int_0^{\infty} r^{\lambda}e^{-r}S_{\lambda}^{n}(r)S_{\lambda}^{n}(r)\,dr=\frac{(2n+\lambda)n![\Gamma(\lambda)]^2}{\Gamma(\lambda+n)}\ .\quad [\text{See I}]$$

Proceeding in an analogous manner we may derive:

$$\int_0^{\infty} r^{\lambda-1}\text{exp.}\{-\tfrac{1}{2}(\lambda_1+\lambda_2)r\}S_{\lambda}^{n}(\lambda_1 r)S_{\lambda}^{m}(\lambda_2 r)dr\qquad \{m\neq n\}$$

$$=\frac{n!m![\Gamma(\lambda)]^2}{\Gamma(\lambda+n)\Gamma(\lambda+m)}\frac{1}{\{\frac{1}{2}(\lambda_1+\lambda_2)\}^{\lambda}}\sum_{\tau=0}^{min(m,n)}(-)^{n-\tau}\frac{\Gamma(\lambda+n+m-\tau)}{\tau!(n-\tau)!(m-\tau)!}\theta^{n+m-2\tau}.$$

$$\ldots\quad (3.8)$$

IV

We have established in I the following general type of integral:

$$\int_0^\infty z^p e^{-(\lambda_1+\lambda_2)z} S_\lambda^n(2\lambda_1 z) S_\mu^m(2\lambda_2 z)\,dz \qquad \{p+1>\lambda>\mu\}$$

$$= \sum_{\beta=0}^{n} \sum_{\gamma=0}^{m} \sum_{\delta=0}^{min(m\ n)} (-)^{m+n+\gamma}\,\theta^{\beta+\gamma} \frac{n\,!m\,!\Gamma(\lambda)\Gamma(\mu)}{(\lambda_1+\lambda_2)^{p+1}\Gamma(\lambda+n)\Gamma(\mu+m)}$$

$$\times \binom{p+1-\lambda}{n-\beta-\delta}\binom{p+1-\mu}{m-\gamma-\delta}\frac{\Gamma(p+1+\beta+\gamma+\delta)}{\beta\,!\gamma\,!\delta\,!}. \qquad \ldots \quad (4.1)$$

Putting $\beta+\delta=\tau$, $\gamma+\delta=\tau'$, $\delta=\nu$, the right-hand member can be rewritten as

$$= \sum_{\tau=0}^{n} \sum_{\tau'=0}^{m} \sum_{\nu=0}^{min(m,n)} (-)^{m+n+\tau'-\nu}\,\theta^{\tau+\tau'-2\nu} \frac{n\,!m\,!\Gamma(\lambda)\Gamma(\mu)}{(\lambda_1+\lambda_2)^{p+1}\Gamma(\lambda+n)\Gamma(\mu+m)}$$

$$\times \binom{p+1-\lambda}{n-\tau}\binom{p+1-\mu}{m-\tau'}\frac{\Gamma(p+1+\tau+\tau'-\nu)}{\nu\,!(\tau-\nu)\,!(\tau'-\nu)\,!}.$$

Similarly, we can derive the following:

$$\int_0^\infty z^p e^{-(\lambda_1+\lambda_2)z} S_\lambda^n(2\lambda_1 z) S_\mu^m(2\lambda_2 z) \qquad \{\lambda>p+1>\mu\}$$

$$= \sum_{\beta=0}^{n} \sum_{\gamma=0}^{m} \sum_{\delta=0}^{min(m,n)} (-)^{m+\beta+\gamma+\delta}\,\theta^{\beta+\gamma} \frac{n\,!m\,!\Gamma(\lambda)\Gamma(\mu)}{(\lambda_1+\lambda_2)^{p+1}\Gamma(\lambda+n)\Gamma(\mu+m)}$$

$$\times \binom{p+1-\mu}{m-\gamma-\delta}\frac{\Gamma(\lambda-p-1+n-\beta-\delta)}{(n-\beta-\delta)\,!\Gamma(\lambda-p-1)}\ \frac{\Gamma(p+1+\beta+\gamma+\delta)}{\beta\,!\gamma\,!\delta\,!}$$

$$= \sum_{\tau=0}^{n} \sum_{\tau'=0}^{m} \sum_{\nu=0}^{min(m,n)} (-)^{m+\tau+\tau'-\nu}\,\theta^{\tau+\tau'-2\nu} \frac{n\,!m\,!\Gamma(\lambda)\Gamma(\mu)}{(\lambda_1+\lambda_2)^{p+1}\Gamma(\lambda+n)\Gamma(\mu+m)}$$

$$\times \binom{p+1-\mu}{m-\tau'}\frac{\Gamma(\lambda-p-1+n-\tau)\Gamma(p+1+\tau+\tau'-\nu)}{(n-\tau)\,!(\tau-\nu)\,!(\tau'-\nu)\,!\nu\,!}. \qquad \ldots \quad (4.2)$$

A corresponding integral for $\lambda>\mu>p+1$ can be evaluated.

DEPARTMENT OF APPLIED MATHEMATICS,
CALCUTTA UNIVERSITY.

Reference

Basu, K , 1943, *Bull. Cal. Math. Soc.*, 35, 21.

ON A PROBLEM IN HEAT CONDUCTION IN A SPHERE

By

N. R. Sen

(Received November 26, 1949)

Recent investigations of stellar models include the case of a gas sphere of which the central core is in isothermal condition, and there is generation of energy within a very thin shell just surrounding the core. The energy generated then flows through an external atmosphere by means of radiation; the temperature falls outwards and vanishes along with the density at the surface. It is found that for the known (radiative) law of energy transmission and a steady state of configuration, given the total mass and the temperature of the core, as the ratio x of the radius of the core to the total radius of the gas sphere R increases, the radius R decreases, and attains a minimum for $x \sim 0.15$. For $x > 0.15$, R increases again, but for equilibrium configurations x itself has a maximum for $x \sim 0.16-0.17$. Similarly, the ratio of the mass of the core to the total mass of the sphere has also a maximum beyond which no equilibrium configurations are possible (Henrich and Chandrasekhar, 1941).

We note that when energy is generated on the surface of the isothermal core, and flows outwards through the radiative atmosphere of a gas sphere of given total mass, for variations in the size of the core the given gas mass has a minimum radius. The complicated law of transmission of energy in a gas mass within which both temperature and density vary makes this minimum property somewhat difficult to comprehend. We construct in this paper an analogous solid model of a (heat)-conducting sphere which shows a similar minimum property. The sphere is supposed to have an isothermal core just on the surface of which within a very thin shell heat energy is continuously being generated, say through some chemical action. This heat flows through the body of the sphere outside the core. Moreover the sphere is in a steady thermal state with either its surface kept at a constant temperature (say zero temperature), or its surface at constant temperature radiating heat outside. There is one difference between the gas and the solid sphere. The gas sphere is supposed to have a given mass but a variable core and radius. The mass of a solid sphere of given density is simply given by its radius, and so its mass will also vary with its radius. We shall show that if the conductivity outside the isothermal core varies according to some simple law, for varying relative size of the core, the radius of the solid sphere has a minimum value. This simple case will help us to get an insight into the more complicated problem.

2

If K be the thermal conductivity, and v the temperature, the equation of steady thermal state in a sphere is

$$\frac{d}{dr}\left(\mathrm{K}r^2\frac{dv}{dr}\right)=0. \qquad \dots \quad (1)$$

Let us assume the conductivity is given by the inverse power law

$$\mathrm{K}=\frac{k_0}{r^n}. \qquad \dots \quad (2)$$

Since there is an isothermal core the failure of this law at the centre need not disturb us.

From (1) we have

$$\mathrm{K}r^2\frac{dv}{dr}=-\mathrm{C}. \qquad \dots \quad (3)$$

If the heat generated per unit of time be Q we must have

$$\mathrm{C}=-\left(\mathrm{K}r^2\frac{dv}{dr}\right)_{r=a}=\frac{\mathrm{Q}}{4\pi} \qquad \dots \quad (3a)$$

where a is the radius of the core.

(a) Suppose the surface of the sphere $r=\mathrm{R}$ is kept at zero temperature. Integrating (3) from $r=\mathrm{R}$, the surface of the sphere where $v=0$, to $r=a$ we obtain say for $n \geq 1$

$$v_a=\frac{\mathrm{Q}}{4\pi}\int_a^{\mathrm{R}}\frac{dr}{\mathrm{K}r^2}=\frac{\mathrm{Q}}{4\pi(n-1)k_0}(\mathrm{R}^{n-1}-a^{n-1}). \qquad \dots \quad (4)$$

Putting $r=x\mathrm{R}$, and $a=x_a\mathrm{R}$, we have since $v_a=v_c=$ temperature of the isothermal core

$$v_c=\frac{\mathrm{QR}^{n-1}}{4\pi(n-1)k_0}(1-x_a^{n-1}). \qquad \dots \quad (5)$$

Thus

$$\frac{4\pi(n-1)k_0}{\mathrm{QR}^{n-1}}=1-x_a^{n-1}\leq 1.$$

For given Q we must have

$$\mathrm{R}^{n-1}\geq\frac{4\pi(n-1)k_0v_c}{\mathrm{Q}}. \qquad \dots \quad (6)$$

Now if Δa be the thickness of the shell within which heat is generated, we write

$$\mathrm{Q}=4\pi a^2\epsilon_0\Delta a, \qquad \dots \quad (7)$$

ϵ_0 being the heat generated per unit volume per unit time. We assume $\epsilon_0\Delta a=\epsilon$, the heat generated ultimately per unit area of the surface of the core per unit of time to be constant. Then we have from (5) and (7)

$$\mathrm{R}^{n+1}=\frac{(n-1)k_0v_c}{\epsilon}\cdot\frac{1}{x_a^2(1-x_a^{n-1})}. \qquad \dots \quad (8)$$

Now as x_a increases from 0 to 1, $x_a^2(1-x_a^{n-1})$ has a maximum for a value

$$x_a = \left[\frac{2}{n+1}\right]^{\frac{1}{n-1}}, \qquad \dots \quad (9)$$

and R will then have its minimum value. *Thus as the relative size of the core increases from a small value, the radius of the solid sphere decreases up to a minimum value, after which the radius increases again as the relative size of the core increases further.* In particular, if $n=2$, the sphere will have its minimum size when the radius of the core is ⅔ of the radius of the sphere; for $n=3$, the minimum size occurs when the core radius is about 71 p.c. of the total radius.

(*b*) Let us suppose there is radiation from the surface of the sphere which is in a steady state thermally. Then we have as the outer boundary condition

$$\left(K\frac{dv}{dr} + hv\right)_{r=R} = 0. \qquad \dots \quad (10)$$

We have now

$$v - v_R = \frac{Q}{4\pi}\int_r^R \frac{dr}{Kr^2}. \qquad \dots \quad (11)$$

Assuming the law of conductivity (2) we obtain

$$v - v_R = \frac{Q}{4\pi k_0 (n-1)} \cdot (R^{n-1} - r^{n-1}). \qquad \dots \quad (12)$$

The surface condition (10) gives

$$-\frac{Q}{4\pi R^2} + hv_R = 0. \qquad \dots \quad (10a)$$

From (12) and (10*a*), putting $a = x_a R$, and $Q = 4\pi a^2 \epsilon$ as before, we obtain

$$R^{n+1} = \frac{(n-1)k_0}{h} \cdot \frac{\alpha^2 - x_a^2}{x_a^2(1-x_a^{n-1})}, \qquad \dots \quad (13)$$

where

$$\alpha^2 = \frac{hv_c}{\epsilon}. \qquad \dots \quad (14)$$

The constant α^2 means the ratio of heat radiated per unit area per unit time from a spherical boundary at temperature v_c, and the amount of heat generated per unit area per unit time at the surface of the core. Two cases arise according as α^2 is greater than, or less than unity. An examination of the function on the right hand side of (13) shows that if $\alpha^2 > 1$, then for $0 < x_a \leq 1$, this function has a minimum for a value of x_a in this interval given by the positive root less than unity of the equation (say n an integer > 1)

$$(n-1)x^{n+1} - (n+1)\alpha^2 x^{n-1} + 2\alpha^2 = 0.$$

If on the other hand $\alpha^2 < 1$, the physically admissible values of x_a are $0 < x_a < \alpha$, and in this interval the function is monotone decreasing and has no minimum.

We thus conclude that if the rate of heat generation on the surface of the isothermal core be slower than the rate of radiation from such a surface (if placed in free space), then for variation of the relative size of the isothermal core the radius of the sphere has a minimum. In particular if $n=2$, the radius of the sphere will be minimum when the relative size of the isothermal core is given by that positive root of the equation

$$x^3-3\alpha^2x+2\alpha^2=0$$

which is less than 1.

If on the other hand the rate of heat generation on the surface of the isothermal core is more rapid than the rate of radiation from such a free surface, the radius of the solid sphere will steadily decrease as the relative size of the isothermal core increases; the relative size of the core cannot then increase beyond a limit given by $x_a=\alpha$ when, however, the radius of the whole sphere shrinks to zero.

Department of Applied Mathematics,
Calcutta University.

Reference

L. B. Henrich and S. Chandrasekhar, 1941, Stellar Models with Isothermal Cores, *Astrophys. J.*, 94, 521.

NOTE ON THE UNIQUENESS OF SOLUTION OF PROBLEMS OF THIN PLATES BENT BY NORMAL PRESSURES

By

BIBHUTIBHUSAN SEN

(Received November 17, 1948)

1. *Introduction*

If the axes of x and y be taken at right angles in the undistorted middle surface of a thin plate, the normal displacement w of the plate subjected to transverse loading of intensity $\phi(x, y)$ satisfies the differential equation

$$\nabla^4 w = \frac{\phi(x, y)}{D} = f(x, y), \qquad \dots \quad (1.1)$$

where $\nabla^4 \equiv \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$, and D is the flexural rigidity.

Using the notations of Love (1927), we obtain the edge conditions as

$$\left.\begin{aligned} &w=0,\ \frac{\partial w}{\partial \nu} = 0, \text{ at a clamped edge,} \\ &w=0,\ G=0, \text{ at a supported edge,} \\ &G=0,\ N - \frac{\partial H}{\partial s} = 0, \text{ at a free edge,} \end{aligned}\right\} \qquad \dots \quad (1.2)$$

where ds is an element of the bounding curve of the edge, $d\nu$ an element of outward drawn normal at any point on it, and N, G, H, the line intensities at the edge, of the shear force, the bending couple and the twisting couple, respectively.

Since the differential equation (1.1) and the various edge conditions expressed in terms of w are obtained by certain processes of approximation, the investigation about the uniqueness of solution of (1.1) subject to different edge condition is not without interest. In a previous paper on the subject (Sen, 1933), the expressions for G and H in terms of ds and $d\nu$ as given by Love (*loc. cit.* (20), p. 465) were assumed. Recently Stevenson (1948) has pointed out certain inaccuracies in these expressions. The object of this note is to re-examine the problem with corrected edge conditions.

If σ be the Poisson's ratio and l, m ($\cos\theta$, $\sin\theta$) be the direction cosines of the outward drawn normal ν, we have as in Love.

$$\left.\begin{aligned} G &= -D\left[l^2\left(\frac{\partial^2 w}{\partial x^2}+\sigma\frac{\partial^2 w}{\partial y^2}\right)+m^2\left(\frac{\partial^2 w}{\partial y^2}+\sigma\frac{\partial^2 w}{\partial x^2}\right)\right.\\ &\qquad\left.+2(1-\sigma)lm\frac{\partial^2 w}{\partial x\partial y}\right]\\ &= -D\left[\sigma\nabla^2 w+(1-\sigma)\left(l^2\frac{\partial^2 w}{\partial x^2}+m^2\frac{\partial^2 w}{\partial y^2}+2lm\frac{\partial^2 w}{\partial x\partial y}\right)\right],\\ H &= D(1-\sigma)\left[lm\left(\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 w}{\partial x^2}\right)+(l^2-m^2)\frac{\partial^2 w}{\partial x\partial y}\right],\\ N &= -D\frac{\partial}{\partial\nu}\nabla^2 w,\text{ where }\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}. \end{aligned}\right\}\quad\ldots\quad(1.3)$$

By means of the formulae

$$\left.\begin{aligned} \frac{\partial}{\partial s}&=l\frac{\partial}{\partial y}-m\frac{\partial}{\partial x}, & \frac{\partial l}{\partial s}&=-m\frac{\partial\theta}{\partial s}, & \frac{\partial m}{\partial s}&=l\frac{\partial\theta}{\partial s},\\ \frac{\partial}{\partial\nu}&=l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}, & \frac{\partial l}{\partial\nu}&=-m\frac{\partial\theta}{\partial\nu}, & \frac{\partial m}{\partial\nu}&=l\frac{\partial\theta}{\partial\nu}, \end{aligned}\right\}\quad\ldots\quad(1.4)$$

we find

$$\begin{aligned} &l^2\frac{\partial^2 w}{\partial x^2}+m^2\frac{\partial^2 w}{\partial y^2}+2lm\frac{\partial^2 w}{\partial x\partial y}\\ &=\left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}\right)\left(l\frac{\partial w}{\partial x}+m\frac{\partial w}{\partial y}\right)-\frac{\partial l}{\partial\nu}\frac{\partial w}{\partial x}-\frac{\partial m}{\partial\nu}\frac{\partial w}{\partial y}\\ &=\frac{\partial^2 w}{\partial\nu^2}-\left(l\frac{\partial w}{\partial y}-m\frac{\partial w}{\partial x}\right)\frac{\partial\theta}{\partial\nu}=\frac{\partial^2 w}{\partial\nu^2}-\frac{\partial w}{\partial s}\frac{\partial\theta}{\partial\nu}, \end{aligned}$$

and

$$\begin{aligned} &lm\left(\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 w}{\partial x^2}\right)+(l^2-m^2)\frac{\partial^2 w}{\partial x\partial y}\\ &=\left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}\right)\left(l\frac{\partial w}{\partial y}-m\frac{\partial w}{\partial x}\right)-\frac{\partial l}{\partial\nu}\frac{\partial w}{\partial y}+\frac{\partial m}{\partial\nu}\frac{\partial w}{\partial x}=\frac{\partial^2 w}{\partial\nu\partial s}+\frac{\partial w}{\partial\nu}\frac{\partial\theta}{\partial\nu}. \end{aligned}$$

Using the above results in (1.3), we have from (1.2) the edge conditions as

$$w=0,\quad\frac{\partial w}{\partial\nu}=0,\qquad\text{at a clamped edge,}\quad\ldots\quad(1.5)$$

$$w=0,\quad\sigma\nabla^2 w+(1-\sigma)\left(\frac{\partial^2 w}{\partial\nu^2}-\frac{\partial w}{\partial s}\frac{\partial\theta}{\partial\nu}\right)=0,\text{ at a supported edge,}\quad(1.6)$$

$$\left.\begin{aligned} &\sigma\nabla^2 w+(1-\sigma)\left(\frac{\partial^2 w}{\partial\nu^2}-\frac{\partial w}{\partial s}\frac{\partial\theta}{\partial\nu}\right)=0,\\ &\frac{\partial}{\partial\nu}\nabla^2 w+(1-\sigma)\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial\nu\partial s}+\frac{\partial w}{\partial\nu}\frac{\partial\theta}{\partial\nu}\right)=0. \end{aligned}\right\}\text{ at a free edge.}\quad(1.7)$$

2(*a*) *Clamped Edge*

We shall assume that w and its derivatives up to the fourth order are continuous and one-valued in the simply connected plane region Ω bounded by the closed curve s of the edge. If U and V be two functions having sufficient differentiability, we have from Green's theorem applied to the same region

$$\iint_{\Omega} \left[\mathrm{U}\nabla^2\mathrm{V} - \mathrm{V}\nabla^2\mathrm{U} \right] d\Omega = \int_s \left[\mathrm{U}\frac{\partial \mathrm{V}}{\partial \nu} - \mathrm{V}\frac{\partial \mathrm{U}}{\partial \nu} \right] ds \quad .. \quad (2.1)$$

Putting $\mathrm{V}=\nabla^2\mathrm{U}$ in the above identity we find

$$\iint_{\Omega} \left[\mathrm{U}\nabla^4\mathrm{U} - (\nabla^2\mathrm{U})^2 \right] d\Omega = \int_s \left[\mathrm{U}\frac{\partial}{\partial \nu}(\nabla^2\mathrm{U}) - \nabla^2\mathrm{U}\frac{\partial \mathrm{U}}{\partial \nu} \right] ds \quad ... \quad (2.2)$$

If possible, let w_1 and w_2 be two solutions of the equation (1.1) which satisfy the boundary condition (1.5). Then $\mathrm{U}=w_1-w_2$ will satisfy the equation $\nabla^4\mathrm{U}=0$ throughout the region and the conditions $\mathrm{U}=\frac{\partial \mathrm{U}}{\partial \nu}=0$ on the boundary. Hence from (2.2) we get $\nabla^2\mathrm{U}=0$ throughout the region. Again from Green's theorem we have

$$\int_s \mathrm{U}_1\frac{\partial \mathrm{U}_2}{\partial \nu}\,ds = \iint_{\Omega} \mathrm{U}_1\nabla^2\mathrm{U}_2 d\Omega + \iint_{\Omega} \left[\frac{\partial \mathrm{U}_1}{\partial x}\frac{\partial \mathrm{U}_2}{\partial x} + \frac{\partial \mathrm{U}_1}{\partial y}\frac{\partial \mathrm{U}_2}{\partial y} \right] d\Omega, \quad (2.3)$$

whence on putting $\mathrm{U}_1=\mathrm{U}_2=\mathrm{U}$, we get

$$\int_s \mathrm{U}\frac{\partial \mathrm{U}}{\partial \nu}\,ds = \iint_{\Omega} \mathrm{U}\nabla^2\mathrm{U}d\Omega + \iint_{\Omega} \left[\left(\frac{\partial \mathrm{U}}{\partial x}\right)^2 + \left(\frac{\partial \mathrm{U}}{\partial y}\right)^2 \right] d\Omega \ldots \quad (2.4)$$

Since $\mathrm{U}=\frac{\partial \mathrm{U}}{\partial \nu}=0$ on the boundary and $\nabla^2\mathrm{U}=0$ everywhere, we have from the above

$$\frac{\partial \mathrm{U}}{\partial x}=\frac{\partial \mathrm{U}}{\partial y}=0, \quad \text{or } \mathrm{U}=\text{constant throughout the region.}$$

But $\mathrm{U}=0$ on the boundary. Hence it is zero everywhere inside it, that is, $w_1=w_2$ at every point of the plate.

2(*b*) *Supported and free Edge*

Whatever may be the boundary conditions, we find from the results (1.5), (1.6) and (1.7) that the relation

$$\left.\begin{aligned} &\int_s \left[\frac{\partial}{\partial \nu}(\nabla^2 w) + (1-\sigma)\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial \nu \partial s} + \frac{\partial w}{\partial \nu}\frac{\partial \theta}{\partial \nu} \right) \right] w\,ds \\ &\quad - \int_s \left[\sigma\nabla^2 w + (1-\sigma)\left(\frac{\partial^2 w}{\partial \nu^2} - \frac{\partial w}{\partial s}\frac{\partial \theta}{\partial \nu} \right) \right] \frac{\partial w}{\partial \nu}\,ds = 0, \end{aligned}\right\}$$

is indentically satisfied.

This identity can be rearranged as

$$\left.\begin{aligned}&(1-\sigma)\int_s w\,\frac{\partial}{\partial\nu}(\nabla^2 w)\,ds+\sigma\int_s\left[w\,\frac{\partial}{\partial\nu}(\nabla^2 w)-\nabla^2 w\,\frac{\partial w}{\partial\nu}\right]ds\\&+(1-\sigma)\int_s\left[w\,\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial\nu\partial s}\right)-\frac{\partial^2 w}{\partial\nu^2}\,\frac{\partial w}{\partial\nu}\right]\\&+(1-\sigma)\int_s\left[w\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial\nu}\,\frac{\partial\theta}{\partial\nu}\right)+\frac{\partial w}{\partial s}\,\frac{\partial\theta}{\partial\nu}\,\frac{\partial w}{\partial\nu}\right]ds=0.\end{aligned}\right\}\quad(2.5)$$

The left hand side of the above identity contains four integrals which we shall consider separately. Putting w for U_1 and $\nabla^2 w$ for U_2 in (2.3) we get the *first* integral

$$(1-\sigma)\int_s w\,\frac{\partial}{\partial\nu}(\nabla^2 w)ds$$

$$=(1-\sigma)\iint_\Omega w\nabla^4 w\,d\Omega+(1-\sigma)\iint_\Omega\left[\frac{\partial w}{\partial x}\,\frac{\partial\nabla^2 w}{\partial x}+\frac{\partial w}{\partial y}\,\frac{\partial\nabla^2 w}{\partial y}\right]d\Omega.\qquad(2.6)$$

Putting w for U in (2.2) we find the *second* integral

$$\sigma\int_s\left[w\,\frac{\partial}{\partial\nu}(\nabla^2 w)-\nabla^2 w\,\frac{\partial w}{\partial\nu}\right]ds=\sigma\iint_\Omega[w\nabla^4 w-(\nabla^2 w)^2]d\Omega.\quad\ldots\quad(2.7)$$

On integrating by parts the first part of the third integral we get the whole integral as

$$(1-\sigma)\left[w\,\frac{\partial^2 w}{\partial\nu\partial s}\right]_1^2-(1-\sigma)\int_s\left(\frac{\partial w}{\partial s}\,\frac{\partial^2 w}{\partial\nu\partial s}+\frac{\partial^2 w}{\partial\nu^2}\,\frac{\partial w}{\partial\nu}\right)ds.$$

The integrated portion of the above expression vanishes, when $w=0$ on the boundary as is the case if the edge is clamped or supported. Since, by hypothesis, w is a single valued function with continuous derivatives on the boundary, the integrated portion vanishes even when w is not zero, as the range of integration extends over the whole circuit and hence the limits 1 and 2 coincide. Thus the third integral reduces to

$$-\frac{(1-\sigma)}{2}\int_s\frac{\partial}{\partial\nu}\left[\left(\frac{\partial w}{\partial s}\right)^2+\left(\frac{\partial w}{\partial\nu}\right)^2\right]ds.$$

This can be again put as

$$-\frac{(1-\sigma)}{2}\int_s\frac{\partial}{\partial\nu}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]ds,$$

since

$$\frac{\partial}{\partial\nu}=l\,\frac{\partial}{\partial x}+m\,\frac{\partial}{\partial y}\text{ and }\frac{\partial}{\partial s}=l\,\frac{\partial}{\partial y}-m\,\frac{\partial}{\partial x}.$$

Applying Green's theorem

$$\int\frac{\partial V}{\partial\nu}ds=\iint_\Omega\nabla^2 V\,d\Omega$$

to the above result, we have the reduced *third* integral

$$-\frac{(1-\sigma)}{2}\int_s \frac{\partial}{\partial \nu}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]ds=-\frac{1-\sigma}{2}\iint_\Omega \nabla^2\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]d\Omega$$

$$=-(1-\sigma)\iint_\Omega\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2+2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right.$$

$$\left.+\frac{\partial w}{\partial x}\frac{\partial \nabla^2 w}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial \nabla^2 w}{\partial y}\right]d\Omega \qquad \ldots \quad (2.8)$$

If the first part of the fourth integral is integrated by parts, the *whole* integral reduces to

$$(1-\sigma)\left[w\frac{\partial w}{\partial \nu}\,\frac{\partial \theta}{\partial \nu}\right]_1^2$$

Let the bounding curve be given by the equation $F(x, y)=0$ where $F(x, y)$ is a continuous and single-valued function of the co-ordinates having continuous derivatives on the doundary. Then we have at a point (x, y) on the boundary, $\theta=\tan^{-1}(F_y/F_x)$ and we find that $\frac{\partial \theta}{\partial \nu}$ will be finite and single valued if there be no singular points on the curve. Hence with the restriction imposed on the values of w mentioned above, we shall obtain for integration over the whole circuit of the boundary, the *fourth* integral,

$$(1-\sigma)\int_s\left[w\frac{\partial}{\partial s}\left(\frac{\partial w}{\partial \nu}\,\frac{\partial \theta}{\partial \nu}\right)+\frac{\partial w}{\partial s}\,\frac{\partial \theta}{\partial \nu}\,\frac{\partial w}{\partial \nu}\right]ds=0. \qquad \ldots \quad (2.9)$$

Since the sum of the four integrals on the left hand side of (2.5) is zero, the sum of the right hand side expressions of (2.6), (2.7), (2.8) and (2.9) will also be zero. Collecting all of them we get

$$\iint_\Omega w\nabla^4 w\,d\Omega$$

$$=\iint_\Omega\left[\sigma(\nabla^2 w)^2+(1-\sigma)\left\{\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2+2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right\}\right]d\Omega. \quad (2.10)$$

Now, if possible, let there be two solutions w_1 and w_2 which satisfy the equation (1.1). Then $W=w_1-w_2$ will make $\nabla^4 W=0$ everywhere. Whatever may be the boundary conditions, w_1 and w_2 will satisfy the relation (2.10) and consequently W will satisfy the same. But as $\nabla^4 W=0$, we shall obtain

$$\iint_\Omega\left[\sigma(\nabla^2 W)^2+(1-\sigma)\left\{\left(\frac{\partial^2 W}{\partial x^2}\right)^2+\left(\frac{\partial^2 W}{\partial y^2}\right)+2\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\right\}\right]d\Omega=0,$$

which gives

$$\nabla^2 W=\frac{\partial^2 W}{\partial x^2}=\frac{\partial^2 W}{\partial y^2}=\frac{\partial^2 W}{\partial x\partial y}=0 \qquad \ldots \quad (2.11)$$

everywhere.

In the case of supported edge $W=0$ on the boundary and since in addition we get $\nabla^2 W=0$ everywhere, we shall find, as in the case of a clamped plate, $W=0$ that is, $w_1=w_2$ at every point of the plate.

When the edge is free, W need not be zero on the boundary. In this case, we find from (2.11) that W $(=w_1-w_2)$ may be a linear function of x and y which has the form of a rigid body displacement.

4. *Conclusion*

Supposing that a thin plate bent by normal pressures is bounded by a continuous closed curve with no singularities, we get the following results. In the case of a plate with clamped or supported edge, there is only one value of the normal displacement w expressed as a single-valued continuous function having continuous derivatives up to the fourth order, which satisfies the equation of equilibrium as well as the boundary conditions. In the case of a plate with free edge, two values of w satisfying the equation of equilibrium and the boundary conditions and subject to the same conditions of continuity and differentiability as mentioned above, may differ by a linear function which represents a rigid body displacement.

BENGAL ENGINEERING COLLEGE,
SIBPORE.

References

Love, A. E. H. (1927), *Mathematical Theory of Elasticity*, 489.
Sen, B. (1933), *Phil. Mag.*, 16, 975-979.
Stevenson, A. C. (1943), *Phil. Mag.*, 34, 105-114.

HIGHLY ABUNDANT NUMBERS

By

S. S. Pillai

(Received December 6, 1943)

If the sum of the reciprocals of the rth powers of the divisors of a number is greater than that for any lower number, then the number may be called a highly abundant number of rth order. In other words, N is a highly abundant number of rth order, if

$$\sigma_{-r}(\mathrm{N}) > \sigma_{-r}(\mathrm{N}'),$$

for all $\mathrm{N}' < \mathrm{N}$, where

$$\sigma_{-r}(\mathrm{N}) = \sum_{d/\mathrm{N}} 1/d^r.$$

According to this definition, a highly composite number of Ramanujan is a highly abundant number of order zero. In this paper, I confine my attention to the cases when $r \geqq 1$, and I use the name highly abundant number only when $r \geqq 1$. Hereafter N always denotes a highly abundant number, unless otherwise mentioned. Writing N in the form

$$2^{a_2}.\ 3^{a_3}\ldots\ldots p^{a_p},$$

I prove

$$a_2 \geqq a_3 \geqq \ldots\ldots \geqq a_p,$$

and that

$$a_p = 1,$$

for all highly abundant values of N, except 4 and 36 when $r=1$, and except 4 when $r \geqq 2$. In similarity with highly composite numbers, it is proved that indices near the beginning form a strictly decreasing sequence and that, near the end, groups of equal indices occur, when N is large. When λ is fairly small in comparison with p, then

$$a_\lambda \log \lambda \sim \log p.$$

Further, it is proved that, when a_2 is given, other indices can vary through only three values, and the indices, except a finite number of those near the beginning, can be determined with an error, at most, of unity. But I am not able to show that two successive highly abundant numbers are asymptotically equivalent.

By comparing the tables for highly composite numbers and highly abundant numbers of the first order, we see many highly composite numbers which are not highly abundant numbers of the first order, of which 7560 is the first one. Further, as far as the tables go, we do not find any highly abundant number of the first order which is not highly composite. So, for a long time, I was under the impression that every highly abundant number of rth order is necessarily one of lower order. Now, it is shown that, in a highly abundant number,

$$a_2 \log 2 \sim \log p.$$

Further, in a highly composite number of Ramanujan, we have

$$a_2 \log 2 \sim \log p/\log 2.$$

Hence, it follows that, after a certain limit, no number will be simultaneously highly composite and highly abundant. So, it is clear that the above conjecture is not true in general. Yet the question whether a highly abundant number of second order is necessarily one of first order and all similar questions remain unanswered.

λ, μ, ν, p and P all stand for primes unless otherwise stated. $f(x) \asymp \phi(x)$ means that there are positive constants k and l such that

$$k\phi(x) < f(x) < l\phi(x).$$

2. Let N be a highly abundant number of rth order and be of the form

$$2^{a_2}.\, 3^{a_3} \ldots\ldots p^{a_p},$$

where $a_s \geqq 0$ and $a_p \geqq 1$. If possible, let

$$a_\mu \geqq a_\nu + 1, \quad \text{when} \quad \mu > \nu. \qquad \ldots \quad (1)$$

Further, let $\qquad \mathrm{N}' = \mathrm{N}.\nu/\mu.$

Then $\mathrm{N}' < \mathrm{N}$ and is an integer, for $a_\mu \geqq 1$, from (1). So

$$\sigma_{-r}(\mathrm{N}) > \sigma_{-r}(\mathrm{N}').$$

Therefore

$$\frac{1+\nu^r+\ldots+\nu^{a_\nu.r}}{\nu^{a_\nu.r}} \times \frac{1+\mu^r+\ldots+\mu^{a_\mu.r}}{\mu^{a_\mu.r}}$$

$$> \frac{1+\nu^r+\ldots+\nu^{(a_\nu+1)r}}{\nu^{(a_\nu+1)r}} \times \frac{1+\mu^r+\ldots+\mu^{(a_\mu-1)r}}{\mu^{(a_\mu-1)r}}.$$

That is $\quad \nu^r\{\nu^{(a_\nu+1)r}-1\}.\{\mu^{(a_\mu+1)r}-1\} > \mu^r\{\nu^{(a_\nu+2)r}-1\}.\{\mu^{a_\mu.r}-1\}.$

So

$$\nu^{(a_\nu+2)r}.(\mu^r-1) > \mu^{(a_\mu+1)r}.(\nu^r-1)+\mu^r-\nu^r > \mu^{(a_\mu+1)r}(\nu^r-1).$$

But $a_\nu \leqq a_\mu - 1$, by our supposition. In other words, $a_\nu + 2 \leqq a_\mu + 1$. So

$$\nu^{(a_\mu+1)r}(\mu^r-1) > \mu^{(a_\mu+1)r}(\nu^r-1).$$

That is

$$(\mu/\nu)^{(a_\mu+1)r} < (\mu^r-1)/(\nu^r-1).$$

But $a_\mu + 1 \geqq 2$, since $a_\mu \geqq 1$. So

$$(\mu/\nu)^{2r} < (\mu^r-1)/(\nu^r-1).$$

Consequently

$$0 > \mu^{2r}.\nu^r - \mu^{2r} - \nu^{2r}.\mu^r + \nu^{2r} = (\mu^r-\nu^r)(\mu^r\nu^r-\mu^r-\nu^r).$$

But $\mu^r - \nu^r > 0$, since $\mu > \nu$. So

$$0 > \mu^r\nu^r - \mu^r - \nu^r = (\mu^r-1)(\nu^r-1)-1 \qquad \ldots \quad (1.a)$$

But $\mu \geqq \nu + 1 \geqq 2 + 1$. So

$$(\mu^r-1)(\nu^r-1)-1 \geqq (\mu-1)(\nu-1)-1 \geqq 1. \qquad \ldots \quad (1.b)$$

Therefore the above is impossible; for (1.a) and (1.b) contradict each other. Hence our supposition that $a_\mu \geqq a_\nu + 1$ is wrong. Hence in a highly abundant number

$$a_2 \geqq a_3 \geqq \cdots \geqq a_p \geqq 1. \qquad \ldots \quad (2)$$

3. Let x and y be two prime factors of N, a and b their respective indices and x be less than y. Now we shall find the relation between a and b.

Let $$n = [\log y / \log x] + 1. \qquad \ldots \quad (3)$$

(*i*) Let $$\mathrm{N}' = \mathrm{N}y/x^n.$$

Further, let us assume for the moment that $a \geqq n$. Then $\mathrm{N}' < \mathrm{N}$ and is an integer. So
$$\sigma_{-r}(\mathrm{N}) \geq \sigma_{-r}(\mathrm{N}').$$
Consequently
$$\frac{1+x^r+\ldots+x^{ar}}{x^{ar}} \times \frac{1+y^r+\ldots+y^{br}}{y^{br}} \geq \frac{1+x^r+\ldots+x^{(a-n)r}}{x^{(a-n)r}} \times \frac{1+y^r+\ldots+y^{(b+1)r}}{y^{(b+1)r}}$$
That is $y^r\{x^{(a+1)r}-1\}.\{y^{(b+1)r}-1\} > x^{nr}\{x^{(a-n+1)r}-1\}\{y^{(b+2)r}-1\}$.
Simplifying the above
$$y^{(b+2)r}(x^{nr}-1) > x^{(a+1)r}(y^r-1)+x^{nr}-y^r \geq x^{(a+1)r}(y^r-1), \text{ from (3).}$$
By taking logarithms, we have
$$(b+2)r\log y > (a+1)r\log x - \log\{(x^{nr}-1)/(y^r-1)\}.$$
So $$b \geq (a+1)\log x/\log y - 2 - \log\{(x^{nr}-1)/(y^r-1)\}/r\log y, \qquad \ldots \quad (4)$$
and $$a < (b+2)\log y/\log x - 1 + \log\{x^{nr}-1)/(y^r-1)\}/r\log x. \qquad \ldots \quad (5)$$
If $a \leqq n$, the above is obviously satisfied. Hence the restriction that $a \leqq n$ is unnecessary.

(*ii*) Let $\mathrm{N}' = \mathrm{N}x^{n-1}/y$; and $b \geqq 1$.

Then $\mathrm{N}' < \mathrm{N}$ and is an integer. So
$$\frac{1+x^r+\ldots+x^{ar}}{x^{ar}} \times \frac{1+y^r+\ldots+y^{br}}{y^{br}} > \frac{1+x^r+\ldots+x^{(a+n-1)r}}{x^{(a+n-1)r}} \times \frac{1+y^r+\ldots+y^{(b-1)r}}{y^{(b-1)r}}$$
That is $\quad x^{(n-1)r}\{x^{(a+1)r}-1\}.\{y^{(b+1)r}-1\} > y^r\{x^{(a+n)r}-1\}\{y^{br}-1\}$.
Simplifying the above, we get that
$$y^{(b+1)r}\{x^{(n-1)r}-1\} < x^{(a+n)r}\{y^r-1\} - (y^r - x^{(n-1)r}) < x^{(a+n)r}(y^r-1)$$
from (3). So
$$y^{br} < x^{(a+1)r}.\{(y^r-1)/(x^{(n-1)r}-1)\}.\{x^{(n-1)r}/y^r\}.$$
Taking logarithms, we get that
$$b < (a+1)\frac{\log x}{\log y} + \log\left\{\frac{y^r-1}{y^r}.\frac{x^{(n-1)r}}{x^{(n-1)r}-1}\right\}\Big/ r\log y, \qquad \ldots \quad (6)$$
and $$a > b\frac{\log y}{\log x} - 1 - \log\left\{\frac{y^r-1}{y^r}.\frac{x^{(n-1)r}}{x^{(n-1)r}-1}\right\}\Big/ r\log x. \qquad \ldots \quad (7)$$
If $b=0$, the above is obviously satisfied. Hence (6) and (7) are true without any restriction.

4. Now we shall prove that $a_p = 1$ except for 4 and 36. Let q be the prime next above p. Then from Bertrand's postulate, $q \leqq 2p-1$. Let $\mathrm{N}' = \mathrm{N}q/2p$. Then N' is an integer and is less than N. So

$$\frac{1+2^r+\ldots+2^{a_2 r}}{2^{a_2 r}} \times \frac{1+p^r+\ldots+p^{a_p r}}{p^{a_p r}}$$

$$> \frac{1+2^r+\ldots+2^{(a_2-1)r}}{2^{(a_2-1)r}} \times \frac{1+\ldots+p^{(a_p-1)r}}{p^{(a_p-1)r}} \times \frac{1+q^r}{q^r}.$$

Therefore

$$q^r(2^{(a_2+1)r}-1)(p^{(a_p+1)r}-1) > 2^r p^r (2^{a_2 r}-1)(p^{a_p r}-1)(1+q^r).$$

Simplifying the above

$$p^{(a_p+1)r}\{2^{(a_2+1)r}-q^r(2^r-1)-2^r\}$$

$$< 2^{(a_2+1)r}\{p^r q^r-(q^r-p^r)\}-(2pq)^r-(2^r p^r-q^r) < 2^{(a_2+1)r}.p^r q^r.$$

In other words,

$$p^{a_p r}\{1-(q^r(2^r-1)+2^r)/2^{(a_2+1)r}\} < q^r < (2p)^r. \qquad \ldots \quad (8)$$

Let $a_p \geqq t$. Then from (7), by putting $x=2$ and $y=p$,

$$(a_2+1)\log 2 > t\log p - \frac{1}{r}\log\left\{\frac{p^r-1}{p^r}\cdot\frac{2^{(n-1)r}}{2^{(n-1)r}-1}\right\}$$

$$> t\log p+\log\{(p^r-1)/p^r\}/r.$$

for $$p^r/(p^r-1) > \{2^{(n-1)r}(p^r-1)\}/\{p^r(2^{(n-1)r}-1)\}.$$

So $$2^{(a_2+1)r} > p^{tr}(p^r-1)/p^r = p^{(t-1)r}(p^r-1).$$

Hence when $a_p \geqq 2$, from (8)

$$p^{(a_p-1)r}\left(1-\{4^r-2^r+(2/p)^r\}/\{p^{(t-2)r}(p^r-1)\}\right) < 2^r, \qquad \ldots \quad (9)$$

for $q \leqq 2p$.

Let $p \geqq 5$ and $t \geqq 2$. Then from (9)

$$p^r \leqq p^{(a_p-1)r} < 2^r(p^r-1)/\{p^r-1-4^r+2^r-(2/p)^r\}, \qquad \ldots \quad (10)$$

$$\leqq 2^r(p-1)/\{p-1-4+2-2/p\} \leqq 2^r(5-1)/(5-1-4+2-\tfrac{2}{5}) = 2^r \times \tfrac{5}{2} \leqq 2^r(\tfrac{5}{2})^r = 5^r.$$

So $p < 5$. Hence when $p \geqq 5$,

$$a_p = 1. \qquad \ldots \quad (11)$$

Therefore, if $a_p \geqq 2$, then N is of the form $2^{a_2}.3^{a_3}$.

Let $p=3$ and $a_3 \geqq 3$. Then from (9)

$$3^{(a_3-1)r}\left(1-\{4^r-2^r+(\tfrac{2}{3})^r\}/\{3^r(3^r-1)\}\right) < 2^r.$$

So $$2^r > 3^{(a_3-1)r}\{1-4^r/(9^r-3^r)\} \geqq 3^{(a_3-1)r} \times \tfrac{1}{3}.$$

Hence $$a_3 < 1+\log 2/\log 3+\log 3/(r\log 3) < 3.$$

So $$a_3 \leqq 2, \text{ if } p=3. \qquad \ldots \quad (12)$$

Let $N=2^m.3^2$, and $m \geqq 3$. Put $N'=2^{m-1}.3.5$. Then $N'<N$. So

$$\frac{2^{(m+1)r}-1}{2^{mr}}.\frac{3^{3r}-1}{3^{2r}} > \frac{2^{mr}-1}{2^{(m-1)r}}.\frac{3^{2r}-1}{3^r}\left(1+\frac{1}{5^r}\right),$$

$$(2^{(m+1)r}-1)(3^{3r}-1) > 2^r.3^r(2^{mr}-1)(3^{2r}-1)\left(1+\frac{1}{5^r}\right),$$

$$(\tfrac{6}{5})^r(2^{mr}-1)(3^{2r}-1) < 2^{(m+1)r}.3^r+2^r.3^{3r}-2^r.3^r-2^{(m+1)r}-3^{3r}+1.$$

So
$$(\tfrac{6}{5})^r < \frac{2^{(m+1)r}}{(2^{mr}-1)(3^r+1)}+\frac{3^{3r}(2^r-1)-2^r.3^r+1}{(2^{mr}-1)(3^{2r}-1)}$$

$$< \frac{2^{4r}}{(2^{3r}-1)(3^r+1)}+\frac{3^{2r}(2^r-1)-6^r+1}{(2^{3r}-1)(3^{2r}-1)}.$$

So
$$\frac{6}{5} < \frac{2^4}{(2^3-1)(3+1)}+\frac{3^2(2-1)-6+1}{(2^3-1)(3^2-1)}=\frac{4}{7}+\frac{4}{56}<1,$$

which is absurd. Hence

$$m=2. \qquad \dots \quad (13)$$

Let $N=2^2.3^2$ and $N'=2^3.3$. Then

$$\frac{2^{3r}-1}{2^{2r}}.\frac{3^{3r}-1}{3^{2r}} > \frac{2^{4r}-1}{2^{3r}}.\frac{3^{2r}-1}{3^r}.$$

That is
$$-2^{4r}-2^r.3^{3r}+2^r > -3^{3r}-3^r.2^{4r}+3^r.$$

In other words,

$$16^r(3^r-1) > 27^r(2^r-1)+3^r-2^r > 27^r(2^r-1).$$

Let $r \geqq 2$. Then from the above

$$\left(\frac{9}{8}\right)^r < \frac{1-\dfrac{1}{3^r}}{1-\dfrac{1}{2^r}} \leqq \frac{1-\dfrac{1}{3^2}}{1-\dfrac{1}{2^2}} = \frac{32}{27}.$$

Hence
$$r < \frac{\log(\frac{32}{27})}{\log(\frac{9}{8})} = \frac{\log(1+\frac{5}{27})}{-\log(1-\frac{1}{9})} < \frac{5}{27}\times 9 < 2.$$

So, when $r \geqq 2$, $N \neq 2^2 3^2$; and. $N=2^2.3^2$ is a highly abundant number of the first order by verification. ... (14)

Lastly, let $N=2^{a_2}$, $a_2 \geqq 3$, and $N'=2^{a_2-2}.3$. Then

$$\{2^{(a_2+1)r}-1\}/2^{a_2 r} > \{(2^{(a_2-1)r}-1)/2^{(a_2-2)r}\}\times\{(3^r+1)/3^r\}.$$

So
$$-3^r > -6^r+2^{(a_2+1)r}-2^{2r}.$$

That is
$$2^{(a_2+1)r} < 6^r+4^r+3^r=6^r\{1+(\tfrac{2}{3})^r+(\tfrac{1}{2})^r\}$$
$$\leqq 6^r.\tfrac{13}{6} < 13^r < 2^{4r},$$

Therefore $a_2+1<4$. That is $a_2 \leqq 2$.

Hence when $p=2$,

$$a_2 \leqq 2. \qquad \ldots \quad (15)$$

Now
$$1+\frac{1}{2^r}+\frac{1}{4^r} > 1+\frac{1}{2^r} > 1+\frac{1}{3^r} > 1.$$

So 4 is a highly abundant number. ... (16)

Consequently from (11)—(16), we arrive at the conclusion that, *in a highly abundant number, a_p, the index of the last prime is unity except in 4 and 36 when $r=1$, and except in 4 when $r \geqq 2$.*

Hereafter, we assume that $N > 36$, so that $a_p = 1$.

5. In (5) put $x=\mu$, $y=P$, where P is the prime next above p. Then $P < 2p$, and $a_P = 0$. Then

$$a_\mu < \frac{2\log P}{\log\mu} - 1 + \log\{(\mu^{nr}-1)/(P^r-1)\} \Big/ (r\log\mu) < \frac{2\log P}{\log\mu}.$$

In (7), put $x=\mu$, $y=p$, then $b=a_p=1$. So

$$a_\mu > \frac{\log p}{\log\mu} - 1 - \log\left\{\frac{p^r-1}{p^r}\cdot\frac{\mu^{(n-1)r}}{\mu^{(n-1)r}-1}\right\}\Big/(r\log\mu)$$

$$> \log p/\log\mu - 2.$$

Hence
$$(\log p/\log\mu) - 2 < a_\mu < 2\log P/\log\mu, \qquad \ldots \quad (17)$$

and since $P < 2p$,

$$a_\mu \log\mu = O(\log p), \quad a_\mu \log\mu \neq o(\log p). \qquad \ldots \quad (18)$$

6. Put $x=\lambda$, $y=\mu$ then $a=a_\lambda$ and $b=a_\mu$. Then from (5)

$$(a_\lambda+1)\log\lambda < a_\mu\log\mu + 2\log\mu + \frac{1}{r}\log\{(\lambda^{nr}-1)/(\mu^r-1)\}.$$

So
$$a_\lambda\log\lambda - a_\mu\log\mu < 2\log\mu - \log\lambda + \frac{1}{r}\log\{(\lambda^{nr}-1)/(\mu^r-1)\}$$

$$< 2\log\mu - \frac{1}{r}\log\{\lambda^r(\mu^r-1)/(\lambda^{nr}-1)\}.$$

Now $\lambda^r(\mu^r-1)-(\lambda^{nr}-1) = \lambda^r\{\mu^r-\lambda^{(n-1)r}-1\}+1 > 0$,

for $\mu \geqq \lambda^{n-1}+1$. So

$$\log\{\lambda^r(\mu^r-1)/(\lambda^{(n-1)r}-1)\} > 0.$$

Hence, if $\lambda < \mu$, then

$$a_\lambda\log\lambda - a_\mu\log\mu < 2\log\mu. \qquad \ldots \quad (19a)$$

From (6),

$$a_\mu\log\mu - a_\lambda\log\lambda < \log\lambda + \frac{1}{r}\log\left\{\frac{\mu^r-1}{\mu^r}\cdot\frac{\lambda^{(n-1)r}}{\lambda^{(n-1)r}-1}\right\}$$

$$< \log\lambda + \log 2. \qquad \ldots \quad (19b)$$

Therefore, since $\lambda \leqq \mu$, from (19a) and (19b), we have

$$| a_\lambda \log \lambda - a_\mu \log \mu | < 2 \log \mu. \qquad \dots \quad (20)$$

Now from (18), $$a_\lambda \log \lambda \ddagger o(\log p).$$

Hence when $$\log \mu = o(\log p), \text{ from (20)},$$

$$a_2 \log 2 \sim a_3 \log 3 \sim \cdots \sim a_\mu \log \mu. \qquad \dots \quad (21)$$

7. Put $x=2$, $y=\mu$, then $a=a_2$, $b=a_\mu$. Then from (4) and (6),

$$a_\mu < (a_2+1)\frac{\log 2}{\log \mu} + \log \left\{ \frac{\mu^r - 1}{\mu^r} \cdot \frac{2^{(n-1)r}}{2^{(n-1)r}-1} \right\} \Big/ (r \log \mu), \qquad \dots \quad (22)$$

and $$a_\mu \geqq (a_2+1)\frac{\log 2}{\log \mu} - 2 - \log\{(2^{nr}-1)/(\mu^r - r)\} \big/ (r \log \mu). \qquad \dots \quad (23)$$

Let $$k = [(a_2+1) \log 2/\log \mu], \quad k+\theta = (a_2+1) \log 2/\log \mu,$$

$$\log \left\{ \frac{\mu^r - 1}{\mu^r} \cdot \frac{2^{(n-1)r}}{2^{(n-1)r}-1} \right\} = \eta_1 r \log \mu, \ \log \frac{2^{nr}-1}{\mu^r - 1} = \eta_2 r \log \mu.$$

Then from (22) and (23),

$$k+\theta-2-\eta_2 \leqq a_\mu < k+\theta+\eta_1,$$

$$\eta_1 \leqq \log\left(\tfrac{2}{3}.\tfrac{2}{1}\right)/\log 3 = \log\left(\tfrac{4}{3}\right)/\log 3 < \tfrac{1}{3},$$

and $$\eta_2 \leqq \{ \log (2^n/\mu) + \log (1 - 1/2^n) - \log (1 - 1/\mu) \}/\log p$$

$$\leqq \{ \log (2^n/\mu) - 1/2^n + 1/(\mu - 1) \}/\log \mu$$

$$\leqq \{ \log 2 + \tfrac{1}{4} - \tfrac{1}{8} \}/\log u, \text{ when } \mu \geqq 5,$$

$$\leqq \tfrac{1}{2},$$

with the condition that $\mu \geqq 5$.

When $\mu = 3$,

$$\eta_2 < \log \frac{4^r - 1}{3^r - 1} \Big/ r \log 3 \leqq \log \frac{4-1}{3-1} \Big/ \log 3 < \tfrac{1}{2}.$$

Hence η_1 and η_2 are each less than $\frac{1}{2}$.

So, if $\theta + \eta_1 \geqq 1$, then $\theta - \eta_2 \geqq 0$. Consequently, $k-1 \leqq a_\mu \leqq k+1$.

If $\theta - \eta_2 < 0$, then $\theta + \eta_1 < 1$. So $k-2 \leqq a_\mu \leqq k$.

If $\theta + \eta_1 \leqq 1$ and $\theta - \eta_2 \geqq 0$, then $k-1 \leqq a_\mu \leqq k$.

Hence, *when a_2 is given, the indices of other primes can range along at most three values, and in most cases it will vary between only two consecutive integers.*

8. In this paragraph, we make some calculations which are necessary for the next one. Let

$$\left.\begin{array}{ll} (i)\ \text{P be the prime next above } p, & (ii)\ \lambda \leqq \mu \leqq \nu \leqq \mathrm{P}, \\ (iii)\ t=\pi(\mu) < \mu, \text{ and } t \geqq 3, & (iv)\ \tfrac{5}{4}\log\nu < \mu < 2\log\nu, \\ \text{and} \qquad (v)\ m_\lambda = [(\log\nu)/(t\log\lambda)]+1. & \end{array}\right\} \quad \ldots \quad (24)$$

From (7), since $\lambda \leqq \nu$,

$$(a_\lambda+1)\log\lambda > a_\nu \log\nu - \frac{1}{r}\log\left\{\frac{\nu^r-1}{\nu^r}\cdot\frac{\lambda^{(n-1)r}}{\lambda^{(n-1)r}-1}\right\}$$

$$> a_\nu\log\nu - \log\{(\lambda^{n-1})/(\lambda^{n-1}-1)\} \geqq a_\nu\log\nu - \log 2. \quad \ldots \quad (25)$$

That is $$\left(\frac{1}{\lambda}\right)^{a_\lambda+1} \leqq \frac{2}{\nu^{a_\nu}}. \quad \ldots \quad (26)$$

From (24), $\quad m_\lambda \log\lambda < (\log\nu)/t + \log\lambda \leqq (\log\nu)/t + \log\mu.$

So $$\lambda^{m_\lambda}t < \nu^{1/t}\mu t < \nu^{1/t}\mu^2 \leqq 4\nu^{1/t}(\log\nu)^2. \quad \ldots \quad (27)$$

From (26) and (27), $$\lambda^{m_\lambda} t \Big/ \lambda^{a_\lambda+1} < 8\nu^{1/t}(\log\nu)^2 \Big/ \nu^{a_\nu}. \quad \ldots \quad (28)$$

In (25), let $\nu = p$. Then, since $2p > \mathrm{P}$,

$$(a_\lambda+1)\log\lambda > \log p - \log 2 > \log(\tfrac{1}{2}\mathrm{P}) - \log 2$$

$$= \log \mathrm{P} - 2\log 2. \quad \ldots \quad (29)$$

When $\nu = \mathrm{P}$, from (27) and (29), since $t \geqq 3$,

$$\lambda^{m_\lambda}.\, t \Big/ \lambda^{a_\lambda+1} \leqq 16\mathrm{P}^{\frac{1}{2}}(\log \mathrm{P})^2 \Big/ \mathrm{P}. \quad \ldots \quad (30)$$

Now $$16\,\nu^{\frac{1}{2}}(\log\nu)^2 \Big/ \nu < 1,$$

when $\nu > c$, where c is a constant. Hence, when $c < \nu \leqq \mathrm{P}$,

$$\lambda^{m_\lambda}.\, t \Big/ \lambda^{a_\lambda+1} \leqq 1. \quad \ldots \quad (31)$$

When $\nu \leqq \mathrm{P}$, from (24) and (25),

$$\mu/2^{a_2} < (4\log\nu)/\nu^{a_\nu}$$

and when $\nu = \mathrm{P}$, from (29) and (24),

$$\mu/2^{a_2} < (r\log \mathrm{P})/\mathrm{P}. \quad \ldots \quad (32)$$

Hence, when $c < \nu \leqq \mathrm{P}$, $\qquad \mu/2^{a_2} < \tfrac{1}{2}.$

Similarly, when $c < \nu \leqq \mathrm{P}$, $\qquad 2t\mu/2^{a_2} < 1. \quad \ldots \quad (33)$

From (33), we can choose a constant c_1, so that for all $\mu > c_1$

$$\log\log\mu > 4\log 2 + 2/\nu + 2t\mu/2^{a_2} + 1. \quad \ldots \quad (34)$$

Further from (19) and (20),

$$a_2\log 2 + 2\log 2 > a_\lambda\log\lambda > a_2\log 2 - 2\log\lambda.$$

So $$4.2^{a_2} > \lambda^{a_\lambda} > 2^{a_2}/\lambda^2.$$

Therefore $$2\mu.2^{a_2+1} > \lambda^{a_\lambda+1} > 2^{a_2+1}/2\mu. \qquad \ldots \quad (35)$$

Again $$\log \nu/(t \log \lambda) < m_\lambda < \log \nu/(t \log \lambda) + 1.$$

That is $$\nu^{1/t} < \lambda^{m_\lambda} < \nu^{1/t}\lambda; \quad \nu^{1/t} < 2^{m_2} < \nu^{1/t}.2.$$

So $$\tfrac{1}{2} < \lambda^{m_\lambda}\big/2^{m_2} < \lambda.$$

Therefore $$2^{m_2}/2 < \lambda^{m_\lambda} < \mu.2^{m_2}. \qquad \ldots \quad (36)$$

Hence from (35) and (36),

$$\lambda^{m_\lambda} \big/ \lambda^{a_\lambda+1} < \mu.2^{m_2}.2\mu \big/ 2^{a_2+1} = \mu^2.2^{m_2} \big/ 2^{a_2}. \qquad \ldots \quad (37)$$

9. Throughout this paragraph we assume that $c < \nu \leqq \mathrm{P}$, and $\mu > c_1$, where P is the prime immediately succeeding p. When $k \leqq \lambda$, from (25)

$$a_k \log k > \log \nu/\pi(\mu), \text{ since } \pi(\mu) \geqq 3. \qquad \ldots \quad (38)$$

Now, let

$$\mathrm{N}' = \mathrm{N}\nu.2^{-m_2}.3^{-m_3}\ldots\mu^{-m_\mu}.$$

Then from (22) and (38), N′ < N and is an integer. So

$$\frac{2^{(a_2+1)r}-1}{2^{a_2r}} \cdots \frac{\mu^{(a_\mu+1)r}-1}{\mu^{a_\mu r}}.\frac{\nu^{(a_\nu+1)r}-1}{\nu^{a_\nu r}}$$

$$> \frac{2^{(a_2-m_2+1)r}-1}{2^{(a_2-m_2)r}} \cdots \frac{\mu^{(a_\mu-m_\mu+1)r}-1}{\mu^{(a_\mu-m_\mu)r}}.\frac{\nu^{(a_\nu+2)r}-1}{\nu^{(a_\nu+1)r}}$$

That is

$$\nu^r(\nu^{(a_\nu+1)r}-1)(1-1/2^{(a_2+1)r})\ldots(1-1/\mu^{(a_\mu+1)r})$$

$$> (\nu^{(a_\nu+2)r}-1)(1-1/2^{(a_2-m_2+1)r})\ldots(1-1/\mu^{(a_\mu-m_\mu+1)r})$$

Therefore

$$-\nu^{(a_\nu+1)r}\left\{\Sigma\frac{1}{2^{(a_2+1)r}} - \Sigma\frac{1}{2^{(a_2+1)r}.3^{(a_2+1)r}} + \ldots\right\} - \left\{1-\frac{1}{2^{(a_2+1)r}}\right\}\cdots$$

$$\cdots\left\{1-\frac{1}{\mu^{(a_\mu+1)r}}\right\} > -\nu^{(a_\nu+1)r}\left\{\Sigma\frac{2^{m_2r}}{2^{(a_2+1)r}} - \Sigma\frac{2^{m_2r}.3^{m_3r}}{2^{(a_2+1)r}.3^{(a_3+1)r}} + \ldots\right\}$$

$$-\frac{1}{\nu^r}\left\{1-\frac{2^{m_2r}}{2^{(a_2+1)r}}\right\}\cdots\left\{1-\frac{\mu^{m_\mu r}}{\mu^{(a_\mu+1)r}}\right\}$$

Hence

$$\nu^{(a_\nu+1)r}\left\{\Sigma\frac{2^{m_2 r}-1}{2^{(a_2+1)r}}-\Sigma\frac{2^{m_2 r}.3^{m_3 r}-1}{2^{(a_2+1)r}.3^{(a_3+1)r}}+\dots\right\}$$

$$>\left\{1-\frac{1}{2^{(a_2+1)r}}\right\}\dots\left\{1-\frac{1}{\mu^{(a_\mu+1)r}}\right\}-\frac{1}{\nu^r}\left\{1-\frac{2^{m_2 r}}{2^{(a_2+1)r}}\right\}\dots\left\{1-\frac{\mu^{m_\mu r}}{\mu^{(a_\mu+1)r}}\right\}$$

$$>\left\{1-\frac{1}{2^{(a_2+1)r}}\right\}\dots\dots\left\{1-\frac{1}{\mu^{(a_\mu+1)r}}\right\}\left\{1-\frac{1}{\nu^r}\right\}.$$

Since the number of terms in each Σ cannot exceed t times the number of terms in the previous Σ, from (29), it follows that

$$\Sigma\frac{2^{m_2 r}.3^{m_3 r}-1}{2^{(a_2+1)r}.3^{(a_3+1)r}}-\Sigma\frac{2^{m_2 r}.3^{m_3 r}.5^{m_5 r}-1}{2^{(a_2+1)r}.3^{(a_3+1)r}.5^{(a_5+1)r}}\geq 0. \quad \dots \quad (39)$$

and a similar result holds for two consecutive Σ's. Hence from (39),

$$\Sigma\frac{2^{m_2 r}}{2^{(a_2+1)r}}>\Sigma\frac{2^{m_2 r}-1}{2^{(a_2+1)r}}-\Sigma\frac{2^{m_2 r}.3^{m_3 r}-1}{2^{(a_2+1)r}.3^{(a_3+1)r}}+\dots\dots\dots, \quad \dots \quad (40)$$

But from (37),

$$\Sigma\frac{2^{m_2 r}}{2^{(a_2+1)r}}<\frac{t.2^{m_2 r}.\mu^{2r}}{2^{a_2 r}}, \quad \dots \quad (41)$$

for the number of terms is t. Further from (35),

$$\left\{1-\frac{1}{2^{(a_2+1)r}}\right\}\left\{1-\frac{1}{3^{(a_3+1)r}}\right\}\dots\left\{1-\frac{1}{\mu^{(a_\mu+1)r}}\right\}>\left\{1-\frac{\mu^r}{2^{a_2 r}}\right\}^t \quad (42)$$

From (40)—(42), it follows that

$$\frac{\nu^{(a_\nu+1)r}.t.2^{m_2 r}.\mu^{2r}}{2^{a_2 r}}>\left\{1-\frac{1}{\nu^r}\right\}\left\{1-\frac{\mu^r}{2^{a_2 r}}\right\}^t. \quad \dots \quad (43)$$

When $x<\frac{1}{2}$, $\log(1-x)>-2x.$ ... (44)

So from (32), (43) and (44), we get that

$$(a_\nu+1)r\log\nu+\log t+m_2 r\log 2+2r\log\mu$$

$$>a_2 r\log 2-2/\nu^r-2t\mu^r/2^{a_2 r}.$$

Hence, since from (24), $m_2\log 2<(\log\nu)/t+1$,

we have $a_2\log 2<(a_\nu+1)\log\nu+m_2\log 2+2\log\mu+\log t+2/\nu+2t\mu/2^{a_2}$

$<(a_\nu+1)\log\nu+(\log\nu)/t+1+2\log\mu+\log\mu+\log 2-\log\log\mu$

$+2/\nu+2t\mu/2^{a_2}$; (for $t=\pi(\mu)<2\mu/(\log\mu)$)

$<(a_\nu+1)\log\nu+(\log\nu)/t+3\log\mu-3\log 2$, from (34)

$\leq(a_\nu+1)\log\nu+4\log\log\nu$; from (24), (iv).

For* $t=\pi(\mu) > 5\mu/(6\log\mu) > \frac{5}{6}.\frac{5}{4}\log\nu/\log(\frac{5}{4}\log\nu) > \log\nu/\log\log\nu$.
Hence when $c < \nu \leqq \mathrm{P}$,

$$a_2\log 2 < (a_\nu+1)\log\nu + 4\log\log\nu. \qquad \ldots \quad (45)$$

10. In the following paragraphs, we shall make some important deductions from the previous results. From (7),

$$a_\nu\log\nu < (a_2+1)\log 2 - \frac{1}{r}\log\frac{\nu^r-1}{\nu^r}.\frac{2^{(n-1)r}}{2^{(n-1)r}-1}.$$

$$\leqq a_2\log 2 + \log 2 + \log\left(\frac{3-1}{3}.\frac{2}{2-1}\right)$$

$$= a_2\log 2 + \log(\tfrac{8}{3}) < a_2\log 2 + 1.$$

So

$$a_2\log 2 > a_\nu\log\nu - 1. \qquad \ldots \quad (46)$$

From (45), when $c < \nu \leqq \mathrm{P}$,

$$a_2\log 2 < (a_\nu+1)\log\nu + 4\log\log\nu. \qquad \ldots \quad (47)$$

Put $\nu=p$ in (46), and $\nu=\mathrm{P}$ in (47). Then $a_\nu=1$ in (46) and $a_\nu=0$ in (47).

Therefore

$$\log p - 1 < a_2\log 2 < \log\mathrm{P} + 4\log\log\mathrm{P}$$

$$< \log p + 1 + 4\log\log p, \quad \text{since } \mathrm{P} < 2p.$$

So

$$a_2\log 2 = \log p + O(\log\log p). \qquad \ldots \quad (48)$$

Hence from (21) and (48), when $\log\lambda = o(\log p)$,

$$a_2\log 2 \sim a_3\log 3 \sim \ldots \sim a_\lambda\log\lambda \sim \log p. \qquad \ldots \quad (49)$$

But in a highly composite number (Ramanujan's memoir, formula (86)).

$$a_2\log 2 \sim (\log p)/\log 2. \qquad \ldots \quad (50)$$

Consequently, from (49) and (50), we arrive at the important result that, *after a certain limit, no number can be simultaneously highly composite and highly abundant.*

The first highly composite number which is not highly abundant is 7560, and we see many such numbers comparing the two tables. But as far as the tables go, every highly abundant number of the first order is a highly composite number. That is to say, I have not calculated the first highly abundant number of the first order which is not highly composite, nor any such single number.

11. From (45) and (46), when $c < \nu \leqq \mathrm{P}$,

$$(a_2\log 2)/(\log\nu) + 1/(\log\nu) > a_\nu$$

$$> (a_2\log 2)/(\log\nu) - 1 - 4(\log\log\nu)/(\log\nu). \qquad \ldots \quad (51)$$

Let

$$\mathrm{A} = [(a_2\log 2)/(\log\nu)], \qquad \ldots \quad (52)$$

and $(a_2\log 2)/(\log\nu) + e_\nu$ be an integer, where $-\frac{1}{2} < e_\nu < \frac{1}{2}$. Hence

$$\left.\begin{aligned}(a_2\log 2)/(\log\nu) &= \mathrm{A}+1-e_\nu, \text{ when } e_\nu > 0\\ &= \mathrm{A}-e_\nu, \quad \text{ when } e_\nu < 0\end{aligned}\right\} \qquad \ldots \quad (53)$$

---

* The result that $\pi(x) > 5x/(6\log x)$, when $x > c$, can be proved by elementary methods.

Let $$e_\nu > 0 \quad \text{and} \quad e_\nu > 1/(\log\nu). \qquad \dots \quad (54)$$

Since $\nu > c$, $$(4\log\log\nu)/(\log\nu) < \tfrac{1}{2}. \qquad \dots \quad (55)$$

So from (51)—(53),

$$A+1-e_\nu+1/(\log\nu) > a_\nu > A-e_\nu-(4\log\log\nu)/(\log\nu). \qquad \dots \quad (56)$$

Now from (53)—(55),

$$-e_\nu+1/(\log\nu) < 0, \quad \text{and} \quad -e_\nu-(4\log\log\nu)/(\log\nu) > -1.$$

So $$A+1 > a_\nu > A-1.$$

Hence, since a_ν is an integer, when $e_\nu > 0$ and $e_\nu > 1/\log\nu$,

$$a_\nu = A \qquad \dots \quad (57)$$

Secondly, let $e_\nu < 0$, and $-e_\nu > (4\log\log\nu)/(\log\nu)$.

$$\left.\begin{array}{ll} \text{So} & -e_\nu+1/(\log\nu) < 1; \text{ for } -\tfrac{1}{2} \leqq e_\nu \text{ and } 1/\log\nu < \tfrac{1}{2}, \\ \text{and} & -e_\nu-(4\log\log\nu)/(\log\nu) > 0. \end{array}\right\} \qquad \dots \quad (58)$$

Hence from (51)—(58).

$$A-e_\nu+1/(\log\nu) > a_\nu > A-1-e_\nu-(4\log\log\nu)/(\log\nu). \qquad \dots \quad (59)$$

From (58) and (59), it follows that $A+1 > a_\nu > A-1$. That is, when $e_\nu < 0$, and $-e_\nu > (4\log\log\nu)/(\log\nu)$, we have

$$a_\nu = A. \qquad \dots \quad (60)$$

From (57) and (60), it results that, when $\nu > c$, and $|e_\nu| > (4\log\log\nu)/(\log\nu)$,

$$a_\nu = [(a_2\log 2)/(\log\nu)]. \qquad \dots \quad (61)$$

$$\left.\begin{array}{lll} \text{Lastly, when} & e_\nu > 0 \text{ and } e_\nu < 1/(\log\nu), & \\ & a_\nu = A+1 \text{ or } A, \text{ from (56);} & \\ \text{and when} & e_\nu < 0 \text{ and } -e_\nu < (4\log\log\nu)/(\log\nu), & \\ & a_\nu = A \text{ or } A-1, \text{ from (59).} & \end{array}\right\} \qquad \dots \quad (62)$$

Hence, when $\nu > c$, a_ν is determined in terms of a_2 exactly in most cases and with an error never exceeding unity in the other cases. From the results of §7, when $\nu < c$, we know that a_ν can be calculated in terms of a_2 with an error never exceeding one in most cases and with an error never exceeding two in the rest.

12. Let λ and μ be two consecutive odd primes so that $\lambda+2 \leqq \mu \leqq 2\lambda$. From (7),

$$a_\lambda > a_\mu . \frac{\log\mu}{\log\lambda} - 1 - \left\{\log\frac{\mu^r-1}{\mu^r} . \frac{\lambda^{(n-1)r}}{\lambda^{(n-1)r}-1}\right\} \Big/ (r\log\lambda)$$

$$> a_\mu\log(\lambda+2)/(\log\lambda) - 2 = a_\mu + (a_\mu/\log\lambda)\log(1+2/\lambda) - 2. \qquad \dots \quad (63)$$

Now $$\log(1+2x) > x, \text{ if } x \leqq \tfrac{1}{2}. \qquad \dots \quad (64)$$

and $$a_\mu > (a_2\log 2)/(\log\mu) - 2 \geq (a_2\log 2)/(\log 2\lambda) - 2 \text{ from (19}a) \qquad \dots \quad (65)$$

Hence from (63) and (64)

$$a_\lambda \geq a_\mu + \frac{1}{\lambda\log\lambda}\left\{\frac{a_2\log 2}{\log 2\lambda} - 2\right\} - 2.$$

So $a_\lambda > a_\mu$, if $\{1/(\lambda\log\lambda)\}\{(a_2\log 2)/(\log 2\lambda) - 2\} - 2 > 0$,

or if $$2\lambda\log\lambda\log 2\lambda < a_2\log 2 - 2\log 2\lambda,$$

or if $$2\lambda(\log\lambda)^2\left\{1 + \frac{\log 2}{\log\lambda} + \frac{1}{\lambda\log\lambda} + \frac{\log 2}{\lambda(\log\lambda)^2}\right\} < a_2\log 2.$$

Now $$\frac{\log 2}{\log\lambda} + \frac{1}{\lambda\log\lambda} + \frac{\log 2}{\lambda(\log\lambda)^2} < \tfrac{1}{2},$$

if $\lambda \geqq 7$. So when $\lambda \geqq 7$,

$$a_\lambda > a_\mu, \text{ if } (\lambda\log\lambda)^2 < (a_2\log 2)/3. \qquad \dots \quad (66)$$

But, by verification, if $\lambda(\log\lambda)^2 < (a_2\log 2)/3$, then $a_\lambda > a_\mu$ when $\lambda \leqq 5$ (including the case when $\lambda = 2$).

Further $\lambda(\log\lambda)^2 < (a_2\log 2)/3$, if $\lambda \leq (a_2\log 2)/3(\log a_2)^2$.

Hence, if $\lambda < (a_2\log 2)/3(\log a_2)^2$, then

$$a_2 > a_3 \geq \dots\dots a_\lambda > a_\mu. \qquad \dots \quad (67)$$

Or, in other words, *in a large highly abundant number, the indices near the beginning form a strictly decreasing sequence.*

13. Let us write N in the form

$$\begin{array}{l} 2.\ \ 3.\ \ 5 \dots\dots p_1 \\ \times 2.\ \ 3.\ \ 5 \dots\dots p_2 \\ \times 2.\ \ 3.\ \ 5 \dots\dots p_3 \\ \times \dots\dots\dots\dots\dots\dots \end{array} \qquad \dots \quad (68)$$

where $p_1 > p_2 \geqq p_3 \geqq \dots\dots$, and the number of rows is a_2. Let P_n be the prime next above p_n, so that $\log P_n = \log p_n + O(1)$. Then (Ramanujan's memoir, paragraph 9)

$$a_{p_n} \geqq n \text{ and } a_{P_n} \leqq n-1, \text{ and so } a_{P_n} \leqq a_{p_n} - 1. \qquad \dots \quad (69)$$

Put $\nu = P_n$ in (45) and $\nu = p_n$ in (46). Then from (69), it follows that

$$n\log p_n - 1 < a_{p_n}\log p_n - 1 < a_2\log 2$$

$$\leqq (a_{P_n} + 1)\log P_n + 4\log\log P_n \leqq n\log P_n + O(\log\log P_n)$$

$$= n\log p_n + O\{n(P_n - p_n)/p_n\} + O(\log\log p_n). \qquad \dots \quad (70a)$$

So $$n\log p_n = a_2\log 2 + O\{n(P_n - p_n)/p_n\} + O(\log\log p_n) \qquad \dots \quad (70)$$

$$= \log p_1 + O\{n(P_n - p_n)/p_n\} + O(\log\log p_1), \text{ from (48)}. \qquad \dots \quad (71)$$

From Bertrand's postulate $(P_n - p_n)/p_n = O(1)$. So

$$n\log p_n = \log p_1 + O(n) + O(\log\log p_1).$$

Hence $$n\log p_n = \log p_1 + O(\log\log p_1) \qquad \dots \quad (72)$$

if $n=O(\log\log p_1)$, and if $n \neq o(\log\log p_1)$,

$$n\log p_n=\log p_1+O(n). \qquad \dots \quad (73)$$

From (72) and (73), it follows that if $n=o(\log p_1)$, then

$$n\log p_n \sim \log p_1. \qquad \dots \quad (74)$$

Let us assume the prime number theorem. Then $(P_n-p_n)/p_n=o(1)$. So from (71),

$$n\log p_n=\log p_1+o(n)+O(\log\log p_1).$$

Hence
$$n\log p_n \sim \log p_1, \text{ for all } n. \qquad \dots \quad (75)$$

14. From (70a),

$$a_{p_n}\log p_n \leqq n\log p_n+O\left\{\frac{n(P_n-p_n)}{p_n}\right\}+O(\log\log p_n). \qquad \dots \quad (76)$$

From (72) and (73), $\log p_1=O(n\log p_n)$. So

$$1/(\log p_n)=O\{n/(\log p_1)\}. \qquad \dots \quad (77)$$

From (76) and (77), it follows that

$$a_{p_n}\leqq n+O\left\{\frac{n^2}{\log p_1}\cdot\frac{P_n-p_n}{p_n}\right\}+O(1).$$

But $a_{p_n}\geqq n$ from (69). So

$$a_{p_n}=n+O\left\{\frac{n^2}{\log p_1}\cdot\frac{P_n-p_n}{p_n}\right\}+o(1). \qquad \dots \quad (78)$$

Consequently, in virtue of Bertrand's postulate,

$$a_{p_n}=n+O\{n^2/(\log p_1)\}+o(1). \qquad \dots \quad (79)$$

So, if $n=o(\sqrt{\log p_1})$, then

$$a_{p_n}=n. \qquad \dots \quad (80)$$

But $a_{P_n}\leqq n-1$, from (69). Hence, if $n=o(\sqrt{\log p_1})$, then

$$p_1>p_2>\dots>p_n \qquad \dots \quad (81)$$

In other words, *in a large highly abundant number, groups of equal indices occur near the end.*

If we assume all known results about $\pi(x)$, then from (78), it follows that if $n=O\{(\log p_1)^{1-\epsilon}\}$, $\epsilon>0$, then

$$a_{p_n}=n; \qquad \dots \quad (82)$$

for
$$(P_n-p_n)/p_n=O\{1/(\log p_n)^k\}=O\{n^k/(\log p_1)^k\},$$

Again, if we assume results about $\pi(x)$ derived from Riemann hypothesis, it follows that, if $n=o\{(\log p_1)/(\log\log p_1)\}$, then

$$a_{p_n}=n. \qquad \dots \quad (83)$$

15. Now we shall find the relation between N and p_1. From (68),

$$\log N = \theta(p_1) + \theta(p_2) + \theta(p_3) + \ldots\ldots, \qquad \ldots \quad (84)$$

where $\theta(x) = \sum_{p \leqq x} \log p$. It is known that (Landau, §18, §53)

$$\theta(x) \asymp x, \qquad \text{(elementary)},$$

$$\theta(x) = x + O\{x/(\log x)^t\}, \quad \text{for every } t, \quad \text{(deep)}.$$

So

$$\log N = p_1 + p_2 + p_3 + \ldots + O\{p_1/(\log p_1)^t\}. \qquad \ldots \quad (85)$$

From (70a), $2 \log p_2 - 1 \lesssim a_2 \log 2 < \log P_1 + 4 \log \log P_1$, from (45)

$$= \log p_1 + 4 \log \log p_1 + o(1).$$

So

$$p_2 = O\{\sqrt{p}.(\log p_1)^2\}. \qquad \ldots \quad (86)$$

and $p_3 = o(p_2)$, and so on. Hence from (85) and (86), we get that

$$\log N \asymp p_1, \qquad \text{(elementary)},$$

and

$$\log N = p_1 \mp O\{p_1/(\log p_1)^t\}, \qquad \text{(deep)}.$$

$$\left.\begin{aligned} &\text{So} && \log p_1 = \log \log N + O(1), && \text{(elementary)}, \\ &\text{and} && p_1 = \log N + O\{(\log N)/(\log \log N)^t\}, && \text{(deep)}. \end{aligned}\right\} \qquad \ldots \quad (87)$$

Let us assume the Riemann hypothesis. Then (Landau, p. 388).

$$\theta(x) = x + O(\sqrt{x}.\ \log^2 x).$$

So from (84) and (86), we have $\log N = p_1 + O(\sqrt{p_1}.\log^2 p_1)$.

That is

$$p_1 = \log N + O\{\sqrt{(\log N)}.(\log \log N)^2\}. \qquad \ldots \quad (88)$$

16. Now we shall consider the order of N(x), where N(x) denotes the number of highly abundant numbers of rth order not exceeding x. Since

$$\sigma_{-r}(2n) \gtrsim \sigma_{-r}(n),$$

there is at least one highly abundant number between n and $2n$. Therefore, it follows that

$$N(x) \neq o(\log x).$$

I am not able to improve this further. The problem to find an upper limit for N(x) seems to be as difficult as the corresponding one for highly composite numbers. We shall consider how far we can proceed.

Let us consider M(a_2), where M(a_2) denotes the number of highly abundant numbers in which the power of 2 is a constant a_2. From (62), a_ν is not definite only when $0 < \epsilon_\nu < 1/(\log \nu)$, and $0 < -\epsilon_\nu < (4 \log \log \nu)/(\log \nu)$.

Let $\epsilon_\nu \lesssim 0$ and $-\epsilon_\nu < (4 \log \log \nu)/(\log \nu)$. Then from (53),

$$(a_2 \log 2)/(\log \nu) = A - \epsilon_\nu < A + 4(\log \log \nu)/\log \nu.$$

So

$$a_2 \log 2 \lesssim A \log \nu + 4 \log \log \nu.$$

Hence

$$2^{a_2} \lesssim \nu^A.(\log \nu)^4.$$

But from (46), $\log \nu \lesssim 2a_2 + 1$. So

$$\nu^A \gtrsim 2^{a_2}/(2a_2 + 1)^4. \qquad \ldots \quad (88a)$$

Similarly, if $e_\nu \geq 0$ and $e_\nu \leqq 1/\log \nu$, then

$$\nu^{i+1} \leqq 2^{a_2}.e. \qquad \ldots \quad (88b)$$

So from (62), (88*a*) and (88*b*), it follows that only when

$$2^{a_2}/(a_2 \log 2+1)^4 \leqq \nu^n \leqq 2^{a_2}.e,$$

α_ν is ambiguous. Even then it can vary only between two values. Since $a_\nu \geqq a_\mu$, when $\mu > \nu$, we get at most

$$\pi(2^{a_2}e)^{\frac{1}{n}} - \pi\{2^{a_2}/(a_2 \log 2+1)^4\}^{\frac{1}{n}}$$

combinations when n is given. This cannot exceed $(2^{a_2}.e)^{\frac{1}{n}}$. But the total number of values that n can assume is a_2. Hence

$$M(a_2) < \prod_{n=2}^{a_2} (2^{a_2}.e)^{\frac{1}{n}} \leqq (2^{a_2}.e)^{\log a_2}.$$

But from (45),

$$\begin{aligned} 2^{a_2} &< P(\log P)^4 = O\{p.(\log p)^4\} \\ &= O\{\log N.(\log\log N)^4\}, \quad \text{from (88).} \\ a_2 &< (\log\log N)/\log a_2 + 4l_3(n) + O(1). \\ \log a_2 &< \log\log\log(N) + O(1). \end{aligned}$$

So
$$\begin{aligned} M(a_2) &< O\{p(\log p)^4\}^{\log a_2} = O\{\log N.(\log\log N)^4\}^{l_3(N)+O(1)}. \\ &\leqq (\log N)^{l_3(N)+O(1)}. \end{aligned}$$

Hence
$$\begin{aligned} N(x) &\leqq a_2 M(a_2) \leqq 2\log\log N.(\log N)^{l_3(N)+O(1)} \\ &= (\log N)^{\log\log\log N+O(1)} \end{aligned}$$

In other words,
$$N(x) = O\{(\log N)^{\log\log\log N+O(1)}.\}.$$

Table of highly abundant numbers

2
2^2
2.3
$2^2.3$
$2^3.3$
$2^2.3^2$
$2^4.3$
$2^2.3.5$
$2^3.3.5$
$2^2.3^2.5$
$2^4.3.5$
$2^3.3^2.5$
$2^4.3^2.5$
$2^3.3.5.7$
$2^2.3^2.5.7$

$2^4.3.5.7$
$2^3.3^2.5.7$
$2^4.3^2.5.7$
$2^5.3^2.5.7$
$2^4.3^3.5.7$
$2^4.3^2.5^2.7$
$2^3.3^2.5.7.11$
$2^4.3^2.5.7.11$
$2^5.3^2.5.7.11$
$2^4.3^3.5.7.11$
$2^4.3^2.5^2.7.11$
$2^5.3^3.5.7.11$
$2^5.3^2.5^2.7.11$
$2^6.3^3.5.7.11$
$2^4.3^2.5.7.11.13$

$2^5 3^2.5.7.11.13$
$2^4.3^3.5.7.11.13$
$2^4.3^2.5^2.7.11.13$
$2^5 3^3.5.7.11.13$
$2^5.3^2.5^2.7.11.13$
$2^6.3^3.5.7.11.13$
$2^4.3^3.5^2.7.11.13$
$2^5 3^3.5^2.7.11.13$
$2^4.3^3.5.7.11.13.17$
$2^4.3^2.5^2.7.11.13.17$
$2^5.3^3.5.7.11.13.17$
$2^5.3^2.5^2.7.11.13.17$
$2^6.3^3.5.7.11.13.17$
$2^4 3^3.5^2.7.11.13.17$
$2^5 3^3.5^2.7.11.13.17$

$2^4.3^3.5.7.11.13.17.19$
$2^6.3^3.5^2.7.11.13.17$
$2^5.3^4.5^2.7.11.13.17$
$2^5.3^3.5.7.11.13.17.19$
$2^5.3^2.5^2.7.11.13.17.19$
$2^6.3^3.5.7.11.13.17.19$
$2^4.3^3.5^2.7.11.13.17.19$
$2^6.3^2.5^2.7.11.13.17.19$
$2^6.3^3.5^2.7.11.13.17.19$
$2^5.3^4.5^2.7.11.13.17.19$
$2^7 3^3.5^2.7.11.13.17.19$
$2^6.3^4.5^2.7.11.13.17.19$
$2^5.3^3.5^2.7^2.11.13.17.19$
$2^6.3^3.5.7.11.13.17.19.23$
$2^4.3^3.5^2.7.11.13.17.19.23$

DEPARTMENT OF PURE MATHEMATICS,
CALCUTTA UNIVERSITY.

CALCUTTA UNIVERSITY CENTRAL LIBRARY

Reference

Landau, *Primzahlen.*

BULLETIN

OF THE

CALCUTTA MATHEMATICAL SOCIETY

A limited number of back Volumes of the Bulletin of the Calcutta Mathematical Society will be sold at reduced prices.

For particulars write to :—

THE SECRETARY, CALCUTTA MATHEMATICAL SOCIETY,
92, UPPER CIRCULAR ROAD, CALCUTTA.

PRINTED IN INDIA

PRINTED BY DINABANDHU GANGULEE, B.A., SUPERINTENDENT, CALCUTTA UNIVERSITY PRESS, 48, HAZRA ROAD, BALLYGUNGE, CALCUTTA, AND PUBLISHED BY THE CALCUTTA MATHEMATICAL SOCIETY, AT 92, UPPER CIRCULAR ROAD, CALCUTTA.